Dirac geometry and moment maps

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(notes taken by Sven Porst)

June 5, 2009

I Dirac structures and Dixmier-Douady bundles

DIRAC STRUCTURES & DD-BUNDLES
(joint with Eckhard Meinrenken)

Motivation 1) relate different geometric incarnations of $H^3(M,\mathbb{Z})$

2) Fact: (M,ω) symplectic manifold. $\Rightarrow$ M is Spinc

i.e. there is a Morita morphism $Cl(TM) \dashrightarrow \mathbb{C}$.

Get it by finding compatible almost complex structure

$J: TM \to TM$ $J^2 = -I$ & $g(x,y) = \omega(Jx,y)$ is

positive definite. Get splitting of complexification:

$$T^{\mathbb{C}}M = \underset{+i}{TM^+} \oplus \underset{-i}{TM^-} \qquad Cl(TM) \circlearrowright \Lambda TM^+ = S$$

Clifford module

G = compact, connected, simply connected Lie group

$B: \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ $\mathfrak{g} = \mathrm{Lie}\, G$

$(G \circlearrowright M,\ \Phi: M \to G,\ \omega \in \Omega^2 M)$ is a q-Hamiltonian G-space

1) $d\omega = -\phi^*\eta$ where η = Cartan 3-form on G $= \frac{1}{12} B(\theta^L, [\theta^L, \theta^L])$

2) $\ker\omega \cap \ker d\phi = 0$

Maurer-Cartan form

3) moment map condition

Thm $Cl(TM) \dashrightarrow \Phi^* \mathcal{A}_G^{h^\vee}$ — $h^\vee$ dual Coxeter of G (e.g. $SU(n)$ $h^\vee = n$))

„twisted Spin$_c$-structure" — $\mathcal{A}_G$ standard (generator $1 \in \mathbb{Z} = H^3(G, \mathbb{Z})$) DD-bundle over G.

Dirac Structures

$V \to M$ vector bundle$/\mathbb{R}$ $\quad \mathbb{V} = V \oplus V^*$, scalar product on fibres

$$\langle (v_1, \alpha_1), (v_2, \alpha_2) \rangle = \alpha_1(v_2) + \alpha_2(v_1)$$

$E \subset \mathbb{V}$ subbundle is Dirac if

- $\dim E_m = \dim V$
- $\forall x, y \in E_m$, $\langle x, y \rangle = 0$

„maximal isotropic" or „Lagrangian" subbundle.

DD bundles

$\mathcal{A} \to M$ bundles with fibre $\mathbb{K}(H)$ compact operators on a seperable Hilbert space H.

Ex (M, g) Riemannian, then

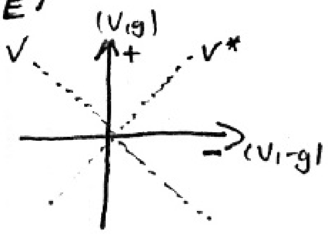
$Cl(TM, g)$ = DD bundle over M.

<u>Idea:</u> Assign to E (Dirac structure) $\longrightarrow$ $\mathcal{A}_E$ (DD-bundle)
in a (more or less) functorial way

<u>Properties</u> (of this assignment)

1) $\mathcal{A}_{V^*} \dashrightarrow \mathbb{C}$ $\qquad$ $\mathcal{A}_V \dashrightarrow Cl(V)$

2) (char classes) $E \longrightarrow \mathcal{A}_E \rightsquigarrow DD(\mathcal{A}_E)$



choose g metric on V.

$\mathbb{V} = V \oplus V^* \simeq (V,g) \oplus (V,-g)$

$E_m \subset \mathbb{V}_m$ is a graph of

$h_m \in O(V,g) \qquad E_m = \{ (v, h_m(v)) \mid v \in V \}$

<u>Assume</u> V trivial bundle

$$E \rightsquigarrow h : M \longrightarrow O(V,g)$$

$$H^3(O(V,g), \mathbb{Z}) \ni [\eta]$$

$$h^*[\eta] \in H^3(M, \mathbb{Z})$$

(generalises to arbitrary bundles)

$$h_E^*[\eta] = DD(\mathcal{A}_E)$$

— relation between classes for DD bundle and from Dirac structure

3)

$$\begin{tikzcd}
(\mathbb{V}, E) \arrow[r, "(\vartheta, \omega)"] \arrow[d] & (\mathbb{V}', E') \arrow[d] \\
M \arrow[r, "\Phi"] & M'
\end{tikzcd}$$

Dirac morphism

$\rightsquigarrow$

$$(\varepsilon, \Phi) : \mathcal{A}_E \dashrightarrow \mathcal{A}_{E'}$$

Morita morphism up to 2-iso

Composition

$$\begin{tikzcd}
 & (\mathbb{V}', E) \arrow[dr, dashed] & \\
(\mathbb{V}, E) \arrow[ur, dashed] \arrow[rr, dashed] & & (\mathbb{V}'', E'')
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{A}_{E'} \arrow[dr, dashed] \arrow[d, phantom, "\Downarrow 2\mathrm{iso}"] & \\
\mathcal{A}_E \arrow[ur, dashed] \arrow[rr, dashed] & {} & \mathcal{A}_{E''}
\end{tikzcd}$$

Examples

1) (M, ω) symplectic, $\mathbb{T}M = TM \oplus T^*M$

$(E_\omega)_m = \{ v + i_v \omega \mid v \in T_m M \}$

$\Downarrow$

$$Cl(TM) \dashrightarrow \mathcal{A}_{TM} = \mathcal{A}_{E_\omega} \dashrightarrow \mathbb{C}$$

is the same as the „standard" Spin_c structure

2) $(M, \phi: M \to G, \omega \in \Omega^2 M)$ q-Hamiltonian space

$$\Updownarrow$$

$$\begin{tikzcd} (\mathbb{T}M, TM) \arrow[r, dashed, "(d\phi, \omega)"] \arrow[d] & (\mathbb{T}G, E_G) \arrow[d] \\ M \arrow[r, "\phi"] & G \end{tikzcd}$$

Dirac morphism

$$Cl(\mathbb{T}M) \longrightarrow \mathcal{A}_{\mathbb{T}M} \longrightarrow \mathcal{A}_{E_G} \simeq \mathcal{A}_G^{L^V}$$

Construction:

1. Choose g, metric on V and define $h_m \in O(V_m)$ s.th. E_m = graph of h_m.
2. Consider $H_m = L^2([0,1], V_m)$ is a real Hilbert space with $\langle f_1, f_2 \rangle = \int_0^1 g_m(f_1(s), f_2(s))\, ds$.
   Consider $\frac{d}{ds}$ $\circlearrowleft$ H_m dense domain.
   boundary conditions s.th. $\frac{d}{ds}$ be anti-selfadjoint
   $E_m \rightsquigarrow f \in$ domain of $\frac{d}{ds}$ s.th. $(f(0), f(1)) \in E_m$
   $f(1) + h_m(f(0)) = 0$ $\qquad (V, g) \oplus (V, -g)$
3. $Cl(H_m)$ $\qquad D_m = \frac{d}{ds}$ with boundary condition E_m
   $\frac{\circlearrowleft}{\Lambda H_m^+} = \mathcal{J}_m$ $\qquad \rightsquigarrow J = i\,\mathrm{sign}\left(\frac{D}{i}\right)$ $\qquad H_m^{\mathbb{C}} = H_m^+ \oplus H_m^-$
   $(\mathcal{A}_E)_m = \mathbb{K}(\mathcal{J}_m)$