The Yang-Mills moduli space on Riemann surfaces

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Introduction

This text deals with some aspects of Atiyah's and Bott's paper [2] on the Yang–Mills equations over Riemann surfaces. First we understand the moduli space of flat connections on a Riemann surface as a symplectic reduction. This moduli space can also be described as a space of representations of the fundamental group.

Finally we look at Atiyah's and Bott's generalization of this correspondence for Yang–Mills connections. This involves a central extension of the fundamental group and its representations.

1 Notation and preliminaries

Let Σ be a connected closed Riemann surface, G a compact connected Lie group, $\mathfrak{g} = Lie(G)$ its Lie algebra and $P \to \Sigma$ a principal G-bundle. By $R_h : P \to P,\ R_h(p) = ph$

we denote the right action. For $\xi \in \mathfrak{g}$ the corresponding fundamental vector field $K^\xi \in \Gamma(TP)$ for this action is

$$K^\xi_p = \frac{\mathrm{d}}{\mathrm{d}t} R_{\exp(t\xi)}(p)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} p \exp(t\xi)|_{t=0}$$

For the action of G on itself by conjugation we write $c\colon G \to \mathrm{Aut}(G),\ c_g(h) = ghg^{-1}$. Its derivative $\mathrm{Ad}_g := T_e c_g \in \mathrm{Aut}(\mathfrak{g})$ defines the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$. The corresponding associated bundles are denoted by $\mathrm{Ad}\, P := P \times_c G$ and $\mathrm{ad}\, P := P \times_{\mathrm{Ad}} \mathfrak{g}$.

The vertical bundle is $\mathscr{V} = \ker(T\pi\colon TP \to \pi^* T\Sigma)$. We get a short exact sequence of vectorbundles

$$0 \to \mathscr{V} \to TP \overset{T\pi}{\to} \pi^* T\Sigma \to 0 \tag{1}$$

Note that $\xi \mapsto K^\xi_p$ is an isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathscr{V}_p$ and thus $\mathscr{V} \cong \underline{\mathfrak{g}}$ is the trivial bundle with fibre $\mathfrak{g}$.

A connection on P is a 1-form $A \in \Omega^1(P, \mathfrak{g})$ on P with values in $\mathfrak{g}$ which satisfies:

$$\begin{aligned} &R_h^* A = \mathrm{Ad}_{h^{-1}} A && \forall h \in G && (\textit{equivariance}) \\ &A(K^\xi) = \xi && \forall \xi \in \mathfrak{g} && (\textit{splitting}) \end{aligned}$$

A connection defines a (equivariant) distribution of horizontal subspaces $\mathcal{H}_p = \ker(A_p)$ and a G-equivariant splitting of the short exact sequence (1), $TP \cong \mathcal{H} \oplus \mathscr{V}$. The restriction of $T\pi$ to $\mathcal{H}$ is an isomorphism $T\pi|_{\mathcal{H}} : \mathcal{H} \xrightarrow{\sim} \pi^* T\Sigma$.

A differential form α on P is said to be horizontal if it vanishes on vertical vectors, i.e.

$$\forall X \in \mathscr{V}\colon \iota_X \alpha = 0$$

The space $\mathcal{A}$ of connections on P is an affine space modelled on

$$\begin{aligned} \Omega^1(P, \mathfrak{g})^G_{hor} =& \{\theta \in \Omega^1(P, \mathfrak{g}) \mid \forall X \in \mathscr{V} : \iota_X \theta = 0,\ \forall h \in G\colon R_h^* \theta = \mathrm{Ad}_{h^{-1}} \theta\} \\ \cong& \Omega^1(\Sigma, \mathrm{ad}\, P) \end{aligned}$$

the space of equivariant horizontal 1-forms on P with values in $\mathfrak{g}$.

The curvature of a connection A is the 2-form $F_A \in \Omega^2(P, \mathfrak{g})^G_{hor} \cong \Omega^2(\Sigma, \mathrm{ad}\, P)$ given by

$$F_A = \mathrm{d}A + \frac{1}{2}[A, A]$$

Furthermore A defines a differential d_A on $\Omega^*(P,\mathfrak{g})^G_{hor} \cong \Omega^*(\Sigma, \operatorname{ad} P)$. This is defined as follows:

$$\begin{aligned} \mathrm{d}_A \colon \Omega^k(P,\mathfrak{g})^G_{hor} &\to \Omega^{k+1}(P,\mathfrak{g})^G_{hor} \\ \alpha &\mapsto \mathrm{d}\alpha + [A,\alpha] \end{aligned}$$

Let $\mathcal{G} = \Gamma(\operatorname{Ad} P) \cong \operatorname{Aut}(P) \cong C^\infty(P,G)^G$ be the gauge group of P. Most of the time we will interpret elements of the gauge group as smooth maps $g : P \to G$, which satisfy the equivariance condition $g(ph) = h^{-1}g(p)h \ \forall h \in G$. The corresponding isomorphism of P is $f(p) = pg(p)$. The Lie algebra of the gauge group is $Lie(\mathcal{G}) = \Gamma(\operatorname{ad} P)$. The gauge group $\mathcal{G}$ acts on the space of connections $\mathcal{A}$ by $(g,A) \mapsto A^g = \operatorname{Ad}_{g^{-1}}(A) + TL_{g^{-1}}Tg$ where $g\colon P \to G$ is a gauge transformation, $Tg\colon TP \to TG$ its differential and $L_h\colon G \to G$ given by $L_h(h') = hh'$. More explicitely $A^g_p = \operatorname{Ad}_{g^{-1}(p)}(A_p) + T_{g(p)}L_{g^{-1}(p)} \cdot T_p g = \operatorname{Ad}_{g^{-1}(p)}(A_p) + g^*\eta_p$ where η is the left-invariant Maurer–Cartan form. This defines a right action of $\mathcal{G}$ on $\mathcal{A}$ denoted by $\varphi_g\colon \mathcal{A} \to \mathcal{A}$ for $g \in \mathcal{A}$.

1.1 Remark Being a space of sections in a bundle the gauge group $\mathcal{G}$ is infinite dimesional. However it is equipped with a Fréchet topology is thus an example for an (infinite dimensional) Fréchet Lie group.

1.2 Lemma *For a gauge transformation $g \in \mathcal{G}$, the curvature F_{A^g} of A^g is*

$$F_{A^g} = \operatorname{Ad}_{g^{-1}}(F_A)$$

Proof. Let $g \in \mathcal{G} = \Gamma(\operatorname{Ad} P)$. Then

$$A^g = \operatorname{Ad}_{g^{-1}} + g^*\eta$$

with the left-invariant Maurer–Cartan form η on G. Hence we get

$$\begin{aligned} (F_{A^g})_p &= \mathrm{d}_p A^g + \frac{1}{2}[A^g_p, A^g_p] \\ &= \mathrm{d}_p\Big(q \mapsto \operatorname{Ad}_{g^{-1}(q)} A_q\Big) + \mathrm{d}_p(g^*\eta) + \frac{1}{2}\Big[\operatorname{Ad}_{g^{-1}(p)} A_p + (g^*\eta)_p, \operatorname{Ad}_{g^{-1}(p)} A_p + (g^*\eta)_p\Big] \\ &= \operatorname{Ad}_{g^{-1}(p)} \mathrm{d}_p A + \mathrm{d}_p\Big(q \mapsto \operatorname{Ad}_{g^{-1}(q)}\Big)A_p + \mathrm{d}_p(g^*\eta) + \frac{1}{2}\Big[\operatorname{Ad}_{g^{-1}(p)} A_p, \operatorname{Ad}_{g^{-1}(p)} A_p\Big] \\ &\quad + \frac{1}{2}\Big[(g^*\eta)_p, (g^*\eta)_p\Big] + \Big[(g^*\eta)_p, \operatorname{Ad}_{g^{-1}(p)} A_p\Big] \\ &= \operatorname{Ad}_{g^{-1}(p)}(F_A)_p + \mathrm{d}_p\Big(q \mapsto \operatorname{Ad}_{g^{-1}(q)}\Big)A_p + \Big[(g^*\eta)_p, \operatorname{Ad}_{g^{-1}(p)} A_p\Big], \end{aligned}$$

using the Maurer–Cartan equation $\mathrm{d}\eta+\frac{1}{2}[\eta,\eta]=0$ in the last step. Consider a smooth curve $q(t)$ in P with $q(0)=p$. Then we have

$$\mathrm{d}_p\Big(q\mapsto \mathrm{Ad}_{g^{-1}(q)}\Big)\cdot\dot{q}(0)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Ad}_{g^{-1}(q(t))}\right)\Big|_{t=0}.$$

We put $g^{-1}(q(t))=g^{-1}(p)h(t)$ with $h(t)=L_{g(p)}g^{-1}(q(t))$, $h(0)=1\in G$, and get $\mathrm{Ad}_{g^{-1}(q(t))}=\mathrm{Ad}_{g^{-1}(p)}\,\mathrm{Ad}_{h(t)}$. Therefore $\dot{h}(0)=\Big((g^{-1})^*\eta\Big)(\dot{q}(0))$. Hence we can continue with

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Ad}_{g^{-1}(p)}\,\mathrm{Ad}_{h(t)}\right)\Big|_{t=0}=\mathrm{Ad}_{g^{-1}(p)}\,\mathrm{ad}_{\dot{h}(0)}=\mathrm{Ad}_{g^{-1}(p)}\left[\Big((g^{-1})^*\eta\Big)(\dot{q}(0)),\,\cdot\,\right].$$

Thus

$$\begin{aligned}\mathrm{d}_p\Big(q\mapsto \mathrm{Ad}_{g^{-1}(q)}\Big)A_p&=\mathrm{Ad}_{g^{-1}(p)}\Big[(g^{-1})^*\eta,A\Big]_p\\&=\Big[\mathrm{Ad}_{g^{-1}}\cdot(g^{-1})^*\eta,\mathrm{Ad}_{g^{-1}}A\Big]_p\\&=-\Big[(g^*\eta)_p,\mathrm{Ad}_{g^{-1}(p)}A_p\Big]\end{aligned}$$

and finally

$$(F_{A^g})_p=\mathrm{Ad}_{g^{-1}(p)}(F_A)_p.$$

□

2 Symplectic Reduction

Following Atiyah–Bott [2] we give a symplectic structure ω on $\mathcal{A}$. Let $A\in\mathcal{A}$ be a connection and $\alpha,\beta\in T_A\mathcal{A}\cong\Omega^1(\Sigma,\mathrm{ad}\,P)$.

$$\omega_A(\alpha,\beta):=\int_\Sigma\langle\alpha\wedge\beta\rangle_{\mathfrak{g}}\tag{2}$$

Here

$$\langle\cdot\wedge\cdot\rangle_{\mathfrak{g}}:\Omega^k(\Sigma,\mathrm{ad}\,P)\times\Omega^l(\Sigma,\mathrm{ad}\,P)\to\Omega^{k+l}(\Sigma,\mathbb{R})$$

is the pairing induced by a Ad-invariant scalar product $\langle\cdot,\cdot\rangle:\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$. Observe that this pairing satisfies

$$\mathrm{d}\langle\phi\wedge\psi\rangle_{\mathfrak{g}}=\langle\mathrm{d}_A\phi\wedge\psi\rangle_{\mathfrak{g}}+(-1)^k\langle\phi\wedge\mathrm{d}_A\psi\rangle_{\mathfrak{g}}\tag{3}$$

for $\phi\in\Omega^k(\Sigma,\mathrm{ad}\,P)$, $\psi\in\Omega^l(\Sigma,\mathrm{ad}\,P)$.

Furthermore there is a Hodge star $\Omega^k(\Sigma, \operatorname{ad} P) \to \Omega^{2-k}(\Sigma, \operatorname{ad} P)$ which is defined pointwise as $(*_x \otimes \mathrm{id}_{\mathfrak{g}})\colon \Lambda^k(T_x^*\Sigma) \otimes \mathfrak{g} \to \Lambda^{2-k}(T_x^*\Sigma) \otimes \mathfrak{g}$ for $x \in \Sigma$ where $*_x\colon \Lambda^k(T_x^*\Sigma) \to \Lambda^{2-k}(T_x^*\Sigma)$ is the usual Hodge star. We again write $*$ for this Hodge star on $\Omega^*(\Sigma, \operatorname{ad} P)$.

By $F\colon \mathcal{A} \to \Omega^2(\Sigma, \operatorname{ad} P) = (\Omega^0(\Sigma, \operatorname{ad} P))^* \cong \mathit{Lie}(\mathcal{G})^*,\ A \mapsto F(A) = F_A$ we denote the curvature of A.

2.1 Lemma

1. *ω is a symplectic form on $\mathcal{A}$.*

2. *The action of $\mathcal{G}$ on $\mathcal{A}$ is hamiltonian with moment map $-F$.*

Proof. As $\mathcal{A}$ is an affine space modelled on $\Omega^1(\Sigma, \operatorname{ad} P)$ and ω_A is independent of the connection A (we identify each $T_A\mathcal{A}$ with $\Omega^1(\Sigma, \operatorname{ad} P)$), ω is closed:

$$\mathrm{d}\omega = 0$$

The non-degeneracy of ω can be proved as follows: given a connection $A \in \mathcal{A}$ and $\alpha \in T_A\mathcal{A}$.

$$\omega_A(\alpha, *\,\alpha) = \int_\Sigma \langle \alpha \wedge *\,\alpha \rangle_{\mathfrak{g}} = \|\alpha\|^2_{L^2(\Lambda^2(T^*\Sigma)\otimes \operatorname{ad} P)} \geq 0$$

and $\omega_A(\alpha, *\,\alpha) = 0$ if and only if $\alpha = 0$. This shows that ω is a symplectic form on $\mathcal{A}$.

Let $A \in \mathcal{A}$ and $\phi \in \mathit{Lie}(\mathcal{G}) \cong \Omega^0(\Sigma, \operatorname{ad} P)$. The fundamental vector field generated by ϕ is

$$\begin{aligned}
K_A^\phi &= \frac{\mathrm{d}}{\mathrm{d}t}\left(A^{\exp(t\phi)}\right)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Ad}_{\exp(-t\phi)} A + \exp(t\phi)^*\eta|_{t=0} \\
&= \operatorname{ad}_{-\phi} A + \frac{\mathrm{d}}{\mathrm{d}t}\left(TL_{\exp(-t\phi)}T\exp(t\phi)\right)|_{t=0} = \operatorname{ad}_{-\phi} A + 0 + TL_{\exp(0)}T\phi \\
&= T\phi - [\phi, A] = \mathrm{d}\phi + [A, \phi] \\
&= \mathrm{d}_A\phi
\end{aligned}$$

Thus we obtain $K_A^\phi = \mathrm{d}_A\phi$.

The induced action of $\mathcal{G}$ on $T\mathcal{A}$ is the following. For $A \in \mathcal{A}$ and $g \in \mathcal{G}$ we get $T_A\varphi_g\colon T_A\mathcal{A} \to T_{A^g}\mathcal{A}$

$$\begin{aligned}
T_A\varphi_g(\alpha) &= \frac{\mathrm{d}}{\mathrm{d}t}(A + t\alpha)^g|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Ad}_{g^{-1}}(A) + t \operatorname{Ad}_{g^{-1}}(\alpha) + g^*\eta|_{t=0} \\
&= \operatorname{Ad}_{g^{-1}}(\alpha)
\end{aligned}$$

Using the Ad-invariance of the scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$, we calculate for $\alpha,\beta\in T_{A^g}\mathcal{A}$:

$$\begin{aligned}\varphi_g^*\omega_A(\alpha,\beta) &= \omega_{A^g}(T_A\varphi_g(\alpha),T_a\varphi_g(\beta)) = \omega_{A^g}(\mathrm{Ad}_{g^{-1}}(\alpha),\mathrm{Ad}_{g^{-1}}(\beta))\\ &= \int\limits_\Sigma \langle \mathrm{Ad}_{g^{-1}}(\alpha)\wedge \mathrm{Ad}_{g^{-1}}(\beta)\rangle_{\mathfrak{g}}\\ &= \int\limits_\Sigma \langle\alpha\wedge\beta\rangle_{\mathfrak{g}}\\ &= \omega_{A^g}(\alpha,\beta)\end{aligned}$$

This shows that $\varphi_g^*\omega=\omega$ so $\mathcal{G}$ acts on $\mathcal{A}$ by symplectomorphisms.
The map $A\mapsto -F_A$ is also $\mathcal{G}$-equivariant by lemma 1.2:

$$F_{A^g}=\mathrm{Ad}_{g^{-1}}(F_A)=\mathrm{Ad}_g^*(F_A)$$

We can now check the moment map condition

$$\begin{aligned}\omega_A(K^\phi,\psi) &= \int\limits_\Sigma \left\langle K^\phi\wedge\psi\right\rangle_{\mathfrak{g}} = \int\limits_\Sigma \langle \mathrm{d}_A\phi\wedge\psi\rangle_{\mathfrak{g}}\\ &= \int\limits_\Sigma \mathrm{d}\,\langle\phi\wedge\psi\rangle_{\mathfrak{g}} - \int\limits_\Sigma\langle\phi\wedge \mathrm{d}_A\psi\rangle_{\mathfrak{g}} && \text{(equation (3))}\\ &= -\int\limits_\Sigma \langle \mathrm{d}_A\psi\wedge\phi\rangle_{\mathfrak{g}} && \text{(Stokes' theorem)}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Sigma \langle -(F_A+td_A\psi)\wedge\phi\rangle_{\mathfrak{g}}\,|_{t=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Sigma \langle -F_{A+t\psi}\wedge\phi\rangle_{\mathfrak{g}}\,|_{t=0}\\ &= \mathrm{d}\left(\int\limits_\Sigma\langle -F\wedge\phi\rangle_{\mathfrak{g}}\right)(\psi)\end{aligned}$$

for $\phi\in Lie(\mathcal{G})\cong\Omega^0(\Sigma,\mathrm{ad}\,P)$ and $\psi\in T_A\mathcal{A}\cong\Omega^1(\Sigma,\mathrm{ad}\,P)$. ☐

2.2 Definition The moduli space $\mathcal{M}(P)$ of flat connections is defined to be the formal symplectic reduction

$$\mathcal{M}(P):=\mathcal{A}/\!/\mathcal{G}=(-F)^{-1}(0)/\mathcal{G}$$

2.3 Remark Both $\mathcal{A}$ and $\mathcal{G}$ are infinite dimensional manifolds, so we have to be careful using Marsden–Weinstein reduction [7]. However the moduli space $\mathcal{M}(P)$ turns out to be finite dimensional.

2.4 Remark Even in the finite dimensional case of symplectic reduction, we can only conclude that the quotient is smooth if 0 is a regular value of the moment map and the group acts freely. Not assuming this, one still obtains a stratified symplectic space (compare [11] in the finite dimensional case).

2.5 Remark The construction above can be generalized to Riemann surfaces with boundary. In this case one also has to restrict to the subgroup of gauge transformations which are trivial at the boundary. This reduction is a smooth but infinite dimensional manifold (see [1] and [3]).

3 Holonomy

The aim of this section is to establish a correspondence between principal G-bundles with flat connections and group homomorphisms $\pi_1(\Sigma) \to G$. This will be established in theorem 3.2 for which a detailed proof can be found in [12]. While we fixed a principal G-bundle in the previous chapters we now have to consider different isomorphism classes of principal G-bundles at the same time.

3.1 Definition Two principal G-bundles $P_1 \to \Sigma, P_2 \to \Sigma$ with flat connections A_1, A_2 respectively are equivalent if there exists an isomorphism $f\colon P_1 \to P_2$ of principal G-bundles over Σ such that $f^*A_2 = A_1$.

The set of equivalence classes is denoted by $\mathcal{M}(\Sigma, G)$.

For a principal bundle with connection A, there is the notion of holonomy (see [6] for details). As the connection is flat, the holonomy along a path only depends on the homotopy type of the path. For a fixed base point $x_0 \in \Sigma$ and a point in the fibre $p_0 \in P_{x_0}$ holonomy in a flat principal bundle can therefore be interpreted as a homomorphism

$$\mathrm{hol}^A_{p_0}\colon \pi_1(\Sigma, x_0) \to G$$

The holonomy of a loop γ with $\gamma(0) = \gamma(1) = x_0$ is given by the equation

$$\tilde{\gamma}(1) = p_0 \mathrm{hol}^A_{p_0}([\gamma])$$

where $\tilde{\gamma}$ is a horizontal lift of γ (with respect to the connection A) satisfying $\tilde{\gamma}(0) = p_0$. For a different lift $p_1 = p_0 h \in P_{x_0},\ h \in G$ of x_0 the corresponding holonomy homomorphisms are related by an inner automorphism of G:

$$\mathrm{hol}^A_{p_0 h} = h^{-1} \mathrm{hol}^A_{p_0} h = c_{h^{-1}} \circ \mathrm{hol}^A_{p_0}$$

As G acts on the space of homomorphisms

$$\begin{aligned} G \times \mathrm{Hom}(\pi_1(\Sigma), G) &\to \mathrm{Hom}(\pi_1(\Sigma), G) \\ (h, \varphi) &\mapsto c(h^{-1}) \circ \varphi \end{aligned}$$

we obtain an element in the quotient

$$\mathrm{hol}^A \in \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

which is independent of the choice of p_0.

For two equivalent flat bundles the isomorphism f can be interpreted as a gauge transformation $g\colon P \to G$ and the equation $A_1 = f^*A_2$ becomes $A_1 = A_2^g$. In this case the holonomy of the connection $A_1 = A_2^g$ is conjugate to the holonomy of the connection A_2:

$$\mathrm{hol}_{p_0}^{A_1} = \mathrm{hol}_{p_0}^{A_2^g} = c_{g^{-1}(p_0)} \circ \mathrm{hol}_{p_0}^{A_2}$$

Therefore we have a well defined map

$$\mathrm{hol}\colon \mathcal{M}(\Sigma, G) \to \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

3.2 Theorem *The map*

$$\begin{aligned} \mathrm{hol}\colon \mathcal{M}(\Sigma, G) &\to \mathrm{Hom}(\pi_1(\Sigma), G)/G \\ [P, A] &\mapsto \mathrm{hol}^A \end{aligned}$$

is a bijection.

Proof. Given an element $h \in \mathrm{Hom}(\pi_1(\Sigma), G)$, we can construct a principal G-bundle as follows: Let $\pi\colon \tilde{\Sigma} \to \Sigma$ be the universal covering. This is a principal $\pi_1(\Sigma)$-bundle. As $\tilde{\Sigma} \to \Sigma$ is a covering, it has a natural flat connection which induces a flat connection A on the associated bundle $\tilde{\Sigma} \times_h G$. First observe that conjugate homomorphisms $\varphi, \varphi' = c_{a^{-1}} \circ \varphi$ give rise to isomorphic bundles:

$$\begin{aligned} \tilde{\Sigma} \times_\varphi G &\xrightarrow{\sim} \tilde{\Sigma} \times_{\varphi'} G \\ [x, g] &\mapsto [x, c_{a^{-1}}(g)] \end{aligned}$$

This isomorphism is also compatible with the induced flat connections. Thus we have a well defined map

$$\begin{aligned} \mathrm{Hom}(\pi_1(\Sigma), G)/G &\to \mathcal{M}(\Sigma, G) \\ [\varphi] &\mapsto [\tilde{\Sigma} \times_\varphi G, A] \end{aligned}$$

which will be an inverse of $\mathrm{hol}\colon \mathcal{M}(\Sigma, G) \to \mathrm{Hom}(\pi_1(\Sigma), G)/G$. Given a principal G-bundle P with a flat connection one can reconstruct its class in $\mathcal{M}(\Sigma, G)$ from $\mathrm{hol}^A \in \mathrm{Hom}(\Sigma, G)/G$ as follows: Take the associated bundle $P \cong \tilde{\Sigma} \times_{\mathrm{hol}} G$ with the induced flat connection. The isomorphism $\tau\colon \tilde{\Sigma} \times_{\mathrm{hol}} G \xrightarrow{\sim} P$ is constructed as follows: First fix a base point $x_0 \in \tilde{\Sigma}$. Given an element $[x, g] \in \tilde{\Sigma} \times_{\mathrm{hol}} G$ take a path $\gamma\colon [0,1] \to \tilde{\Sigma}$ from $\gamma(0) = x_0$ to $\gamma(1) = x$. This projects to a path $\bar{\gamma} = \pi \circ \gamma\colon [0,1] \to \Sigma$. Identify $G \cong P_{\pi(x_0)}$ and define $\tau([x, g]) \in P_{\pi(x)}$ as the parallel transport of $g \in P_{\pi(x_0)}$ along γ. For a loop $[l] \in \pi_1(\Sigma)$ we have $\tau([x \cdot [l], g]) = \tau([x, \mathrm{hol}([l]^{-1})g])$, so τ is well defined and an isomorphism of principal G-bundles. This shows the surjectivity of $\mathrm{hol}\colon \mathcal{M}(\Sigma, G) \to \mathrm{Hom}(\pi_1(\Sigma), G)/G$. On the other hand, given a homomorphism $\varphi \in \mathrm{Hom}(\pi_1(\Sigma), G)$ and $P = \tilde{\Sigma} \times_\varphi G$ with the induced flat connection A, the holonomy of A is again φ. This shows that $\mathrm{hol}\colon \mathcal{M}(\Sigma, G) \to \mathrm{Hom}(\pi_1(\Sigma), G)/G$ is also injective and therefore an isomorphism. □

3.3 Remark In the case that G is simply connected, which implies that all principal G-bundles P are trivial, we can describe the moduli space $\mathcal{M}(\Sigma \times G)$ in definition 2.2 as $\mathrm{Hom}(\pi_1(\Sigma), G)/G$. In general the moduli spaces $\mathcal{M}(P)$ have to be considered for each isomorphism class of G-principal bundles.

3.4 Corollary $\dim(\mathcal{M}(P)) < \infty$

3.5 Remark In the proof of theorem 3.2 we do not use that $\dim(\Sigma) = 2$. The statement is true for any connected manifold. Compare also [8] and [12].

4 Yang–Mills moduli space and representations

In theorem 3.2 we have seen that flat connections on principal G-bundles over Riemann surfaces correspond to homomorphisms $\pi_1(\Sigma) \to G$. In this section we consider Yang–Mills connections which are the local minima of the Yang–Mills functional

$$L_{YM}(A) = \int_\Sigma \langle F_A \wedge * F_A \rangle_{\mathfrak{g}} .$$

Those connections are the solutions to $\mathrm{d}_A * F_A = 0$. Note that the Yang–Mills functional is invariant under gauge transformations since we have

$$\begin{aligned} L_{YM}(A^g) &= \int_\Sigma \langle F_{A^g} \wedge * F_{A^g} \rangle_{\mathfrak{g}} = \int_\Sigma \langle Ad_{g^{-1}} F_A \wedge * Ad_{g^{-1}} F_A \rangle_{\mathfrak{g}} \\ &= \int_\Sigma \langle F_A \wedge * F_A \rangle_{\mathfrak{g}} = L_{YM}(A) \end{aligned}$$

for all $g \in \mathcal{G}$ and $A \in \mathcal{A}$. Of course, flat connections are also Yang–Mills.

Let G be a compact Lie group and M a smooth manifold. We denote the set of (isomorphism classes) of principal G-bundles $P \to M$ and connections A on P with $\mathcal{A} = \mathcal{A}(M,G) = \coprod_P \mathcal{A}(P)$ and define the groupoid $\mathcal{G} = \mathcal{G}(M,G) = \coprod_P \mathcal{G}(P)$ acting on $\mathcal{A}(M,G)$ canonically. If we consider special connections such as flat or Yang–Mills connections, we use the notation $\mathcal{A}_{\text{flat}}(M,G)$, $\mathcal{A}_{\text{YM}}(M,G)$ instead.

FiXme: *notations!*

For a Riemann surface Σ theorem 3.2 can be generalized to the following theorem by Atiyah and Bott [2]. The fundamental group $\pi_1(\Sigma)$ has to be replaced by a certain central $\mathbb{R}$-extension $\Gamma_{\mathbb{R}}$ of $\pi_1(\Sigma)$, which we discuss later (see equation (5)).

4.1 Theorem *Let Σ be a closed Riemann surface of* $\operatorname{genus}(\Sigma) \geq 1$ *and G a compact Lie group. Then there is a bijection*

$$\operatorname{Hom}_{\text{top}}(\Gamma_{\mathbb{R}}, G)/G \to \mathcal{A}_{\text{YM}}/\mathcal{G}$$

between conjugation classes of continuous *homomorphisms and gauge classes of Yang–Mills connections.*

In the case of the sphere S^2 there is a bijection

$$\operatorname{Hom}_{\text{top}}(S^1, G)/G \to \mathcal{A}_{\text{YM}}/\mathcal{G}$$

Instead of following the proof sketched by Atiyah and Bott we present a proof given by Morrison in [9]. The latter can be regarded as an interpolation between theorem 3.2 and a result of Kobayashi ([4], [5]), which characterizes *arbitrary* connections on principal bundles as conjugation classes of continuous homomorphisms from a certain topological group, the so–called *path group*. We start describing the path group.

Let M be a smooth manifold. Consider the set $\widetilde{\Phi}$ of piecewise smooth paths $\gamma\colon I = [0,1] \to M$ with source and target maps $\sigma(\gamma) = \gamma(0)$ and $\tau(\gamma) = \gamma(1)$. Concatenation of two such paths γ, δ with $\sigma(\delta) = \tau(\gamma)$ yields a piecewise smooth path

$$(\delta\gamma)(t) \begin{cases} \gamma(2t), & t \in [0, \frac{1}{2}], \\ \delta(2t-1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

We consider the equivalence relation generated by:

1. $\gamma \sim \gamma \circ \varphi$ for a piecewise smooth increasing function $\varphi\colon I \to I$ with $\varphi(0) = 0$, $\varphi(1) = 1$
2. $\gamma^{-1}\gamma \sim \underline{\gamma(0)}$, where $\underline{\gamma(0)}$ is the constant path
3. $\gamma_1 \dots \gamma_n \sim \delta_1, \dots \delta_n$ for paths $\gamma_1, \dots \gamma_n, \delta_1, \dots, \delta_n$ with $\gamma_j \sim \delta_j$ (with respect to 1., 2.)

In particular we identify paths with common image and orientation. Under this equivalence relation the quotient $\Phi := \widetilde{\Phi}/\sim$ is a groupoid.

We endow it with a topology by choosing a Riemann metric and hence a metric function d_M on M and define

$$d_\Phi([\gamma],[\delta]) := \inf_{\substack{\gamma_1 \in [\gamma] \\ \delta_1 \in [\delta]}} \sup_{t\in[0,1]} d_M(\gamma_1(t),\delta_1(t)).$$

The *topology* on Φ does not depend on the choice of the metric on M. The maps σ, $\tau\colon \Phi \to M$ are well–defined and continuous. Because of

$$d_\Phi([\gamma_2\gamma_1],[\delta_2\delta_1]) \le \max\{\, d_\Phi([\gamma_2],[\delta_2]), d_\Phi([\gamma_1],[\delta_1]) \,\},$$

Φ is a topological groupoid. Hence, for any base point $a \in M$ we have a topological group $\Phi_a^a := \tau^{-1}(a) \cap \sigma^{-1}(a)$. We call Φ_a^a the *path group at a*. If M is connected, the path groups are isomorphic for different base points.

Now we state the theorem announced by Kobayashi:

4.2 Theorem (Kobayashi) *Let M be a connected manifold, G a compact Lie group and $a \in M$. There is a bijection*

$$\mathcal{A}(M,G)/\mathcal{G}(M) \to \mathrm{Hom}_{\mathrm{top}}(\Phi_a^a, G)/G,$$

where $\mathrm{Hom}_{\mathrm{top}}(\Phi_a^a, G)$ *denotes the space of* continuous *group homomorphisms from* Φ_a^a *to G.*

Proof. Let P be a principal G-bundle with connection A. Then parallel transport assigns to any path $\alpha \in \widetilde{\Phi}$ an isomorphism $\mathrm{hol}^A(\alpha)\colon P_{\sigma(\alpha)} \to P_{\tau(\alpha)}$ of G-spaces and $\mathrm{hol}^A(\alpha)$ depends only on the class of α in Φ. Especially, if α is a loop at a, i. e. an element of Φ_a^a and $p \in P_a$, then $\mathrm{hol}^A(\alpha)(p) = p \cdot \mathrm{hol}_p^A(\alpha)$ defines an element $\mathrm{hol}_p^A(\alpha)$ of G. Hence, $\mathrm{hol}_p^A\colon \Phi_a^a \to G$ is a continuous homomorphism of groups. As we have seen in section 3, a gauge equivalent connection A^g with $g \in \mathcal{G}(M)$ yields an equivalent homomorphism $\mathrm{hol}_p^{A^g} = g(p)^{-1}\mathrm{hol}_p^A g(p)$.

Now consider a homomorphism $\rho\colon \Phi_a^a \to G$. The space $\Phi^a := \sigma^{-1}(a)$ of paths starting at a with the projection map $\tau\colon \Phi^a \to M$ is actually a principal Φ_a^a-bundle, $\gamma \in \Phi_a^a$ acting from the right on $\delta \in \Phi^a$ by $\delta \mapsto \delta\gamma$. The associated bundle $P := \Phi^a \times_\rho G$ is a principal G-bundle and depends only on the conjugation class of ρ. In order to define a connection A on P, we construct horizontal lifts for any path $\alpha \in \widetilde{\Phi}$ as follows: Let $\sigma(\alpha) = \alpha(0) = b$, $\tau(\alpha) = \alpha(1) = c$ and $p = [\delta, g] \in P_b$ with a path δ from a to b and an element $g \in G$. We define a path from δ to $\alpha\delta$ in Φ^a by $\widetilde{\alpha}(t) := \alpha_t\delta$ with α_t the path in Φ given by the class of $s \mapsto \alpha(st)$ in Φ. Indeed, $\widetilde{\alpha}(0) = \delta$ and $\tau(\widetilde{\alpha}(t))$ is a point on α.

Hence $t \mapsto [\widetilde{\alpha}(t), g]$ is a lift of α which starts at p. Furthermore, $[\widetilde{\alpha}(1), g]$ depends only on the equivalence class of α in Φ. Thus, we can define $\mathrm{hol}^A(\alpha)[\delta, g] := [\widetilde{\alpha}(1), g] \in P_c$. For $p = [\underline{a}, 1] \in P_a$, where $\underline{a} \equiv a$ denotes the constant path, and α a loop at a (i. e. $\alpha \in \Phi^a_a$) we have

$$p \cdot \mathrm{hol}^A_p(\alpha) = [\widetilde{\alpha}(1), 1] = [\alpha\underline{a}, 1] = [\underline{a}\alpha, 1] = [\underline{a}, \rho(\alpha)] = p \cdot \rho(\alpha),$$

hence $\mathrm{hol}^A_p(\alpha) = \rho(\alpha)$. Choosing another $p \in P$ yields a homomorphism equivalent to ρ. □

4.3 Remark The (path) connected component $\Phi^a_{a,0}$ of the identity in Φ^a_a consists of classes of (piecewise smoothly) contractible paths. The quotient $\Phi^a_a/\Phi^a_{a,0}$ is isomorphic to the fundamental group $\pi_1(M, a)$. theorem 3.2 characterises $\mathcal{A}_{\text{flat}}(M, G)/\mathcal{G}(M)$ as those homomorphisms $\rho\colon \Phi^a_a \to G$, which are *locally constant* and hence factor through $\Phi^a_a/\Phi^a_{a,0} \cong \pi_1(M, a)$. Indeed $\Phi^a/\Phi^a_{a,0} \cong \widetilde{M}$, the universal covering space.

We see that very special connections might correspond to homomorphisms which are constant on certain normal subgroups H of Φ^a_a. These homomorphisms would be well defined on the quotient group Φ^a_a/H. Since Yang–Mills connections are special, but less special than flat ones, one might hope that their corresponding homomophisms are constant on some subgroub of $\Phi^a_{a,0}$. Surprisingly, this is true on Riemann surfaces:

Let Σ be a closed Riemann surface with K"ahler form ω and a base point $a \in \Sigma$. From now on we always normalize ω such that $\int_\Sigma \omega = 1$. By $\Phi^a_{a,\omega}$ we denote the subgroup of contractible paths γ in Φ^a_a which satisfy

$$\int_{I^2} h^*\omega = 0$$

for some continuous null–homotopy $h\colon I^2 \to \Sigma$ with $h|\partial I^2 = \gamma$ that is smooth on the interior of I^2.

4.4 Theorem *Let Σ be a closed Riemann surface and G a compact Lie group. Then holonomy induces a bijection*

$$\mathcal{A}_{\text{YM}}/\mathcal{G} \to \mathrm{Hom}_{\text{top}}(\Phi^a_a/\Phi^a_{a,\omega}, G)/G.$$

Proof. Let $P \to \Sigma$ be a principal G-bundle and A a Yang–Mills connection. Choose a point $p \in P_a$ and define the holonomy bundle

$$P' := \{\, q \in p \mid \exists \text{ horiz. path } \gamma \text{ s. t. } \gamma(0) = p, \gamma(1) = q \,\} \tag{4}$$

which is a smooth principal bundle with structure group $G' = \mathrm{hol}_p^A(\Phi_a^a)$ (see [6] for details). The group G' is *the holonomy group of A with reference point p.* The bundle P' is a reduction of P and $A' := A|P'$ is a G'-connection and induces a horizontal distribution $\mathcal{H}^{A'} \subset TP'$. One can regain P as the associated bundle $P' \times_{G',\iota} G$ with the inclusion $\iota\colon G' \hookrightarrow G$.

The Ambrose–Singer theorem states that G' has Lie algebra

$$\mathfrak{g}' = \operatorname{span}\left\{ F_{A'}(X,Y) \;\middle|\; X, Y \in \mathcal{H}_q^{A'}, q \in P' \right\}$$

We interpret the map $f := *F_A$ as a section of $\operatorname{ad} P$, or equivalently as a G-equivariant map from P to $\mathfrak{g}$ (equivariant means $f(pg^{-1}) = \mathrm{Ad}_g(f(p))$, $g \in G$, $p \in P$). Because $\mathrm{d}_A f = 0$, it is covariantly constant which implies that $f|P'$ is actually constant. Hence, there is $\xi \in \mathfrak{g}$ such that $f|P' = \xi$ and $F_A = \xi \otimes \widetilde{\omega}$, where $\widetilde{\omega}$ denotes the lift of ω to P'. Because of

$$\mathfrak{g}' = \operatorname{span}\left\{ \xi \otimes \widetilde{\omega}(X,Y) \;\middle|\; X, Y \in \mathcal{H}_q^{A'}, q \in P' \right\} = \mathbb{R}\xi,$$

$\mathfrak{g}'$ is abelian and G' is one–dimensional. Therefore, we set $A' = \xi \otimes \psi$ with a G'-invariant $\psi \in \Omega^1(P', \mathbb{R})$ and $F_{A'} = \mathrm{d}A' = \xi \otimes \mathrm{d}\psi$.

Let $\gamma \in \Phi_{a,0}^a$ be a contractible loop contained in the contractible open set U. Then $P'|U \cong U \times G'$ is trivial. We choose a local trivialisation on U, such that p is mapped to $(a, 1)$. In this local trivialisation $A' = \varphi + \xi \otimes \eta$, where φ denotes the Maurer–Cartan form on G' and η a 1-form on Σ with $\mathrm{d}\eta = \omega$. The horiziontal lift Γ of γ with $\Gamma(0) = p$ has the form $\Gamma(t) = (\gamma(t), g(t))$ with some function $g\colon I \to G'$ and satisfies the equations

$$g(0) = 1, \qquad 0 = \langle A'_{\Gamma(t)}, \dot{\Gamma}(t) \rangle = \xi \cdot \langle \eta_{\gamma(t)}, \dot{\gamma}(t) \rangle + \langle \varphi_{g(t)}, \dot{g}(t) \rangle \,.$$

We conclude that

$$g(t) = \exp\left(\xi \cdot \int_0^t \langle \eta_{\gamma(s)}, \dot{\gamma}(s) \rangle \, \mathrm{d}s \right)$$

and that

$$\mathrm{hol}_p^{A'}(\gamma) = g(1) = \exp\left(\xi \cdot \int_I \gamma^* \eta \right).$$

Since $\mathrm{d}\eta = \omega$ (as $F_{A'} = \xi \otimes \omega = \xi \otimes \mathrm{d}\eta$ in the chosen trivialisation), $\mathrm{hol}_p^{A'}(\gamma) = 1$ if γ is an element of $\Phi_{a,\omega}^a$. Thus, $\mathrm{hol}_p^{A'}\colon \Phi_a^a \to G'$ yields a homomorphism $\rho'\colon \Phi_a^a/\Phi_{a,\omega}^a \to G'$. So the conjugation class of $\iota\rho'$ corresponds to the gauge class of (P, A), which shows that the map $\mathcal{A}_{\mathrm{YM}}/\mathcal{G} \to \mathrm{Hom}_{\mathrm{top}}(\Phi_a^a/\Phi_{a,\omega}^a, G)/G$ is well–defined.

Now let $\rho\colon \Phi_a^a/\Phi_{a,\omega}^a \to G$ be a continuous homomorphism. Since $\Phi_{a,0}^a/\Phi_{a,\omega}^a$ is a one–dimensional Lie group (see lemma 4.5), the image of ρ in G is a one–dimensional Lie group with Lie algebra generated by $\xi \in \mathfrak{g}$. Let $\kappa\colon \Phi_a^a \to \Phi_a^a/\Phi_{a,\omega}^a$ be the quotient map. Because of theorem 4.2 the conjugation class of $\rho\kappa$ defines a principal G-bundle

P with connection A. There is a $p \in P_a$, such that $\mathrm{hol}_p^A = \rho\kappa$. It follows that the holonomy group of $G' = \mathrm{hol}_p^A(\Phi_a^a) = \rho(\Phi_a^a/\Phi_{a,\omega}^a)$ is one-dimensional and that its Lie algebra $\mathfrak{g}' = \mathbb{R}\xi$.

Take an arbitrary $x \in \Sigma$ and $q \in P'$ over x (compare with equation (4)). Let δ be a path whose horizontal lift connects p to q. For X, $Y \in T_x\Sigma$ let γ_t be the loop around the parallelogramm spanned by $\sqrt{t}X$, $\sqrt{t}Y$ in a local Darboux coordinate system. The loop γ_t is contractible, hence an element of $\Phi_{x,0}^x \cong \Phi_{a,0}^a$. We have

$$(F_A)_q(\widetilde{X}, \widetilde{Y}) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{hol}_q^A(\gamma_t)\Big|_{t=0},$$

where $\widetilde{X}$ and $\widetilde{Y}$ denote the horitontal lifts of X and Y in T_qP. On the one hand,

$$\mathrm{hol}_q^A(\gamma_t) = \mathrm{hol}_q^A\delta(\delta^{-1}\gamma_t\delta)\delta^{-1} = \mathrm{hol}^A(\delta)\mathrm{hol}_p^A(\delta^{-1}\gamma_t\delta)\mathrm{hol}^A(\delta^{-1}) = \mathrm{hol}_p^A(\delta^{-1}\gamma_t\delta),$$

since $\mathrm{hol}^A(\delta)\colon P_a \to P_x$ is a G-equivariant diffeomorphism and $\mathrm{hol}_p^A(\delta^{-1}\gamma_t\delta) \in G$. On the other hand, $\mathrm{hol}_p^A(\delta^{-1}\gamma_t\delta) = \exp(\xi\,\mathrm{area}(\delta^{-1}\gamma_t\delta))$ depends only on the "area" enclosed by $\delta^{-1}\gamma_t\delta$ (see lemma 4.5 for the definition of area) which is the same as $\mathrm{area}(\gamma_t) = \omega(\sqrt{t}X, \sqrt{t}Y) = t\omega(X,Y)$, since we chose γ_t to be a parallelogramm in a local Darboux coordinate system. We get

$$(F_A)_q(\widetilde{X}, \widetilde{Y}) = \frac{\mathrm{d}}{\mathrm{d}t}\exp\Big(t\xi \cdot \omega(X,Y)\Big)\Big|_{t=0} = \xi \cdot \omega(X,Y),$$

for all $q \in P(p)$, hence $F_A = \xi \otimes \widetilde{\omega}$ on $P(p)$ and $(F_A)_{qg} = \mathrm{Ad}_{g^{-1}}\,\xi \otimes \widetilde{\omega}_{qg}$ with $g \in G$. It follows that $*F_A$ is convariantly constant, hence $\mathrm{d}_A * F_A = 0$. Indeed, ρ corresponds to a Yang–Mills connection. ☐

4.5 Lemma *If Σ is a closed Riemann surface of* $\mathrm{genus}(\Sigma) \geq 1$ *then* $\Phi_{a,0}^a/\Phi_{a,\omega}^a \cong \mathbb{R}$. *If $\Sigma = S^2$ then $\Phi_{a,0}^a/\Phi_{a,\omega}^a \cong S^1$.*

Proof. Let Σ be a closed Riemann surface of $\mathrm{genus}(\Sigma) \geq 1$. We construct a surjective group homomorphism $\mathrm{area}\colon \Phi_{a,0}^a \to \mathbb{R}$. For any contractible loop γ at a, there is a null–homotopy, hence a continuous map $h\colon I^2 \to \Sigma$ with $h|\partial I^2 = \gamma$, which is smooth in the interior of I^2. We define

$$\mathrm{area}(\gamma) := \int_{I^2} h^*\omega.$$

We might think of it as the area contained in γ although the *interior* of a loop is not that easy to define: For our purpose we need that $\mathrm{area}(\gamma^{-1})$ equals $-\mathrm{area}(\gamma)$ and not $\mathrm{area}\,\Sigma - \mathrm{area}(\gamma)$, which would be the outcome if we defined the interior of γ by orientation.

We have to show now that area is well–defined. Let $k\colon I^2 \to \Sigma$ be another continuous map, smooth in the interior. Gluing h and k together along ∂I^2 and reversing orientation of k yields a map $H\colon S^2 \to \Sigma$. It is smooth when we are not on the equator. We get

$$\int_{I^2} h^*\omega - \int_{I^2} k^*\omega = \int_{S^2} H^*\omega = \deg H \int_\Sigma \omega = \deg H.$$

The fundamental covering space of Σ is contractible for genus$(\Sigma) \geq 1$. So $\pi_k(\Sigma) = 1$ for $k \geq 2$. Our map H is contractible and has zero degree. This shows that area is well–defined. The kernel of area is $\Phi^a_{a,\omega}$, hence we get $\Phi^a_{a,0}/\Phi^a_{a,\omega} \cong \mathbb{R}$.

In the case of the sphere S^2 that H may have arbitrary integer degree, so area(γ) is only well–defined modulo $\mathbb{Z}$. Hence, $\Phi^a_{a,0}/\Phi^a_{a,\omega} \cong S^1$. ☐

We now introduce a central extension $\Gamma_\mathbb{R}$ of $\pi_1(\Sigma)$ to describe $\Phi^a_a/\Phi^a_{a,\omega}$. The fundamental group of a Riemann surface Σ of genus g is given by

$$\pi_1(\Sigma) = \left\{ a_i, b_i, i = 1, \dots, g \;\middle|\; \prod_{i=1}^g [a_i, b_i] = 1 \right\}$$

where a_i and b_i are the generators and $[a_i, b_i]$ is the commutator $a_i b_i a_i^{-1} b_i^{-1}$. Setting

$$\Gamma = \left\{ J, a_i, b_i, i = 1, \dots, g \;\middle|\; [a_i, J] = [b_i, J] = 1, \prod_{i=1}^g [a_i, b_i] = J \right\}$$

we obtain a central extension of $\pi_1(\Sigma)$ by $\mathbb{Z}$:

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z} = \langle J \rangle \arrow[r] & \Gamma \arrow[r] & \pi_1(M) \arrow[r] & 1 \end{tikzcd}$$

Alternatively we can think of Γ as $\mathbb{Z} \times_\omega \pi_1(\Sigma)$ where ω is a cocycle in $H^2(\pi_1(\Sigma), \mathbb{Z}) \subset H^2(\pi_1(\Sigma), \mathbb{R})$(in group cohomology, see also [10]), i. e. with the product $(m, \alpha) \cdot_\omega (n, \beta) = (m + n + \omega(\alpha, \beta), \alpha\beta)$. We use the same cocycle ω to define the group

$$\Gamma_\mathbb{R} = \mathbb{R} \times_\omega \pi_1(\Sigma) \tag{5}$$

which yields the following commutative diagram:

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & \pi_1(M) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbb{R} \arrow[r] & \Gamma_\mathbb{R} \arrow[r] & \pi_1(M) \arrow[r] & 1 \end{tikzcd}$$

Notice that Γ is normal in $\Gamma_\mathbb{R}$. By snake lemma applied to the previous diagram we have

$$\begin{tikzcd} 1 \arrow[r] & S^1 \arrow[r, "\sim"] & \Gamma_\mathbb{R}/\Gamma \arrow[r] & 1 \end{tikzcd}$$

Thus we can identify $\Gamma_{\mathbb{R}}/\Gamma \cong S^1$ and obtain the following exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z} \arrow[r] & \Gamma_{\mathbb{R}} \arrow[r, "\xi_0"] & S^1 \times \pi_1(\Sigma) \arrow[r] & 1 \end{tikzcd}$$

4.6 Lemma *If Σ is a closed Riemann surface of* $\operatorname{genus}(\Sigma) \geq 1$*, then $\Phi^a_a/\Phi^a_{a,\omega} \cong \Gamma_{\mathbb{R}}$. For $\Sigma = S^2$, $\Phi^a_a/\Phi^a_{a,\omega}$ is isomorphic to S^1.*

Proof. For $\Sigma = S^2$ we observe that any loop is contractible, hence $\Phi^a_a = \Phi^a_{a,0}$. For $\operatorname{genus}(\Sigma) \geq 1$, we recall

$$\left(\Phi^a_a/\Phi^a_{a,\omega}\right)/\left(\Phi^a_{a,0}/\Phi^a_{a,\omega}\right) \cong \Phi^a_a/\Phi^a_{a,0} \cong \pi_1(\Sigma, a) \quad \text{and} \quad \Phi^a_{a,0}/\Phi^a_{a,\omega} \cong \mathbb{R}.$$

Thus, we have the exact sequence

$$0 \to \mathbb{R} \to \Phi^a_a/\Phi^a_{a,\omega} \to \pi_1(\Sigma, a) \to 1$$

and see that $\Phi^a_a/\Phi^a_{a,\omega}$ is a central $\mathbb{R}$-extension of $\pi_1(\Sigma, a)$. Let $a_1, \cdots, a_g$, $b_1, \ldots, b_g$ be $2g$ generators of $\pi_1(\Sigma, a)$. They satisfy the relation $\prod_{j=1}^g [a_j, b_j] = 1$ in $\pi_1(\Sigma, a)$. The surface Σ can be obtained from a polygon D with $4g$ egdes labeled

$$a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}$$

by gluing the edges together according to their labels such that all the vertices are glued together to the base point a. The quotient map $h\colon D \to M$ can be seen as a homotopy of $\prod_{j=1}^g [a_j, b_j]$ to $\underline{a}$, which is smooth and injective in the interior of D. Since $\Sigma \setminus h(D^\circ)$ is a set of measure zero, we have

$$\operatorname{area}\Big(\prod_{j=1}^{g} [a_j, b_j]\Big) = \int_{D^\circ} h^*\omega = \int_\Sigma \omega = 1,$$

Hence, $\prod_{j=1}^g [a_j, b_j] = J$ in $\Phi^a_{a,0}/\Phi^a_{a,\omega} \cong \Gamma_{\mathbb{R}}$. □

These lemmata and theorem 4.4 proof theorem 4.1.

For the case of genus zero we note that any continuous homomorphism from S^1 to some topological group G can be seen as a periodic homomorphism from $\mathbb{R} \cong \Gamma_{\mathbb{R}}$ to G.

Finally, we see that every principal bundle over a compact Riemann surface which admits a Yang–Mills connection is associated to the principal $\Phi^a_a/\Phi^a_{a,\omega}$-bundle $\Phi^a/\Phi^a_{a,\omega} \to \Sigma$.

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