FREE RESOLUTIONS OF FAT POINTS ON A CONIC IN $\mathbb{P}^2$

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ABSTRACT. This paper is concerned with free resolutions of fat points in $\mathbb{P}^2$, respectively in some blown up surface of $\mathbb{P}^2$. A short overview is given about methods developed mainly by Harbourne, working with linear systems of divisors on a blown up surface. Finally, these methods are used to compute free resolutions in concrete terms for two cases: for the case that all points lie on a smooth conic and the highest multiplicity occurs four times, and for the case that all but one point lie on a line, and the one point has multiplicity one.

1. INTRODUCTION

We want to compute free resolutions of fat point ideals in $\mathbb{P}^2$.

Let $p_1, \ldots, p_n$ be points in $\mathbb{P}^2$. Associate to every point p_i a multiplicity m_i. By a fat point ideal, we denote an ideal I of codimension 2 in $R := K[x, y, z]$, the coordinate ring of $\mathbb{P}^2$, where I contains all curves passing through the given points with the given multiplicities, i.e. $I = \mathfrak{m}_{p_1}^{m_1} \cap \ldots \cap \mathfrak{m}_{p_n}^{m_n}$, where $\mathfrak{m}_{p_i}$ denotes the vanishing ideal of p_i. (Later on, we will also allow infinitely near points.)

Denote by h_I the Hilbert function of an ideal I. Certainly, the Hilbert function is related to the free resolution of I. We will see that the knowledge of the Hilbert function and some other properties of I will indeed be sufficient to describe the free resolution. If we blow up $\mathbb{P}^2$ at the n points, we get a rational surface X whose divisor class group is generated by $E_0, E_1, \ldots, E_n$, where E_0 denotes the pullback of a line in $\mathbb{P}^2$ and E_i denotes the exceptional divisor of the point p_i for $i = 1, \ldots, n$. These generators fulfill the relations $E_0^2 = 1$, $E_i^2 = -1$ for $i > 0$ and $E_i.E_j = 0$ for $i \neq j$. Then

$$h_I(d) = h^0(X, \mathcal{O}_X(dE_0 - m_1E_1 - \ldots - m_nE_n)$$

(see 3.2). Also, we will see that the other properties of I that are needed in order to give the free resolution can be understood in terms of linear systems of divisors on the blown up surface X. So most of the work that needs to be done is about linear systems on rational surfaces.

Most people working on fat point ideals are rather interested in points in general position. However, for points in general position, the method of blowing up and working with linear systems of divisors does not in general give a solution to the problem. But if we put the points in a special position, the method allows to compute free resolutions. Harbourne showed in [Har98] that a free resolution can be calculated if the points lie on a conic.

In this paper, we want to give a short overview about Harbourne's method. We will use the theory to calculate a free resolution explicitly for two different cases: For the case that $n-1$ points lie on a line, and the n-th point has multiplicity 1, and for the case that all points lie on a smooth conic, and the highest multiplicity occurs four times. The results are presented in the following Theorems A and B. We will give a detailed proof of Theorem B, referring to [Mar03] for the proof of Theorem A, which works analogously.

Theorem A. *Let $I = I(\mathcal{K}, \underline{m})$ be a fat point ideal. Let $(p_1, \dots, p_n)$ be a representative such that $m_1 \geq \dots \geq m_n$. Assume $m_n = 1$. Let p_1 lie on a line $L \subset \mathbb{P}^2$ and let p_i lie on the strict transform of L in $\mathrm{Bl}_{p_{i-1}} \dots \mathrm{Bl}_{p_1} \mathbb{P}^2$ for all $i = 2, \dots, n-1$. (For an explanation of this notation, see chapter 2.) Then the free resolution I is:*

if $m_1 > m_2$ and $\mu_{m_2} = 1$:

$$0 \longrightarrow R[-(m_1+1)]^{m_1-m_2-1} \oplus R[-(m_1+2)]^2 \oplus \bigoplus_{i=1}^{m_2} R[-(a_i+1)] \longrightarrow$$

$$R[-m_1]^{m_1-m_2} \oplus R[-(m_1+1)]^2 \oplus \bigoplus_{i=1}^{m_2} R[-a_i] \longrightarrow I \longrightarrow 0\,;$$

if $m_1 > m_2$ and $\mu_{m_2} > 1$:

$$0 \longrightarrow R[-(m_1+1)]^{m_1-m_2-1} \oplus R[-(m_1+2)] \oplus \bigoplus_{i=1}^{m_2} R[-(a_i+1)] \longrightarrow$$

$$R[-m_1]^{m_1-m_2} \oplus R[-(m_1+1)] \oplus \bigoplus_{i=1}^{m_2} R[-a_i] \longrightarrow I \longrightarrow 0\,;$$

if $m_1 = m_2$ and $\mu_{m_2} = 1$:

$$0 \longrightarrow R[-(m_1+2)]^2 \oplus \bigoplus_{i=1}^{m_2} R[-(a_i+1)] \longrightarrow$$

$$R[-(m_1+1)]^3 \oplus \bigoplus_{i=1}^{m_2} R[-a_i] \longrightarrow I \longrightarrow 0\,;$$

if $m_1 = m_2$ and $\mu_{m_2} > 1$:

$$0 \longrightarrow R[-(m_1+2)] \oplus \bigoplus_{i=1}^{m_2} R[-(a_i+1)] \longrightarrow$$

$$R[-(m_1+1)]^2 \oplus \bigoplus_{i=1}^{m_2} R[-a_i] \longrightarrow I \longrightarrow 0\,;$$

where a_i denotes the special degrees of I, $a_i = m_1 + \mu_{m_2} + \dots + \mu_i$. (For a definition of special degree and μ_i, see page 11 resp. definition 8.)

Theorem B. *Let $I = I(\mathcal{K}, \underline{v})$ be a fat point ideal. Asume that the highest multiplicity, m, i.e. the highest entry in the vector $\underline{v}$, occurs four times in $\underline{v}$. Choose a representative $(p, q, r, s, p_1, \dots, p_n)$ in $\mathcal{K}$ such that the highest multiplicity m is associated to p, q, r and s and such that for the other associated multiplicities, $m_1 \geq \dots \geq m_n$ holds. Assume that p, q, r, s and $p_1, \dots, p_n$ lie on an irreducible conic, respectively on the strict transform of the conic. (For more explanations, see 2 and 5.) Then the free resolution of the ideal I:*

(1) *If $c := m_1 - m_2$ is even, then the free resolution is*

$$0 \longrightarrow R[-(2m+2)]^{m-m_1-1} \oplus \bigoplus_{a=2}^{c/2+1} R[-(2m+a)]^3 \oplus R[-(2m+c/2+2)]$$

$$\oplus \bigoplus_{i=0}^{m_2-1} \left(R[-(2m+\lceil \tfrac{a_{m_2-i}}{2} \rceil +1)]^{j_i} \oplus R[-(2m+\lceil \tfrac{a_{m_2-i}}{2} \rceil +2)]^{l_i} \right) \longrightarrow$$

$$R[-2m]^{m-m_1+1} \oplus \bigoplus_{a=1}^{c/2} R[-(2m+a)]^3 \oplus \bigoplus_{i=0}^{m_2-1} R[-(2m+\tfrac{a_{m_2-i}}{2})]^{k_i} \longrightarrow I \longrightarrow 0.$$

(2) *If $c := m_1 - m_2$ is odd, then the free resolution is:*

$$0 \longrightarrow R[-(2m+2)]^{m-m_1-1} \oplus \bigoplus_{a=2}^{(c-1)/2+1} R[-(2m+a)]^3 \oplus R[-(2m+(c+1)/2+1)]^3$$

$$\oplus \bigoplus_{i=0}^{m_2-1} \left(R[-(2m+\lceil \tfrac{a_{m_2-i}}{2} \rceil +1)]^{j_i} \oplus R[-(2m+\lceil \tfrac{a_{m_2-i}}{2} \rceil +2)]^{l_i} \right) \longrightarrow$$

$$R[-2m]^{m-m_1+1} \oplus \bigoplus_{a=1}^{(c-1)/2} R[-(2m+a)]^3 \oplus R[-(2m+(c+1)/2)]^2$$

$$\oplus \bigoplus_{i=0}^{m_2-1} R[-(2m+\tfrac{a_{m_2-i}}{2})]^{k_i} \longrightarrow I \longrightarrow 0\,;$$

where

$$(k_i, l_i, j_i) := \begin{cases} (1,1,0) & \text{if } a_{m_2-i} \text{ is even} \\ (2,0,2) & \text{if } a_{m_2-i} \text{ is odd} \end{cases}$$

and $a_{m_2-i} = c + i + 1 + \mu_{m_2} + \ldots + \mu_{m_2-i}$. *(For a definition of* a_i *and* μ_i*, see page 11 resp. definition 8 or page 12.)*

2. Infinitely near points

As already said in the introduction, we wish to allow infinitely near points, too. That is, we choose $p_1 \in \mathbb{P}^2$, $p_2 \in \mathrm{Bl}_{p_1}\mathbb{P}^2$ (where $\mathrm{Bl}_{p_1}\mathbb{P}^2$ denotes the blow up of $\mathbb{P}^2$ at the point p_1) and so on. Let $(m_1, \ldots, m_n)$ be the corresponding multiplicities. If the points were all in $\mathbb{P}^2$, we could enumerate them in such a way that $m_1 \geq \ldots \geq m_n$. As we allow infinitely near points, reordering does not make any sense, since the points live in different surfaces. But later on, we need the multiplicities to be ordered. If, for example, $p_1 \in \mathbb{P}^2$ and $p_2 \in \mathrm{Bl}_{p_1}\mathbb{P}^2 \setminus E_1$, it does not really make a difference if we consider the tuple (p_1, p_2) or (p_2, p_1). If $m_2 > m_1$, we would then choose the second possibility. So we try to group tuples of points to equivalence classes and we try to find a representative in each equivalence class where the points are ordered in such a way that $m_1 \geq \ldots \geq m_n$. Let $\pi_i : \mathrm{Bl}_{p_i} \ldots \mathrm{Bl}_{p_1}\mathbb{P}^2 \to Bl_{p_{i-1}} \ldots \mathrm{Bl}_{p_1}\mathbb{P}^2$ denote the map of blowing up the point p_i. Then two tuples $(p_1, \ldots, p_n)$ and $(p'_1, \ldots, p'_n)$ will be called equivalent if there exists an isomorphism $\Phi : \mathrm{Bl}_{p_n} \ldots \mathrm{Bl}_{p_1}\mathbb{P}^2 \to \mathrm{Bl}_{p'_n} \ldots \mathrm{Bl}_{p'_1}\mathbb{P}^2$ such that the following diagram commutes:

$$\begin{array}{ccccc}
 & \mathrm{Bl}_{p_n} \ldots \mathrm{Bl}_{p_1}\mathbb{P}^2 & \xrightarrow{\Phi} & \mathrm{Bl}_{p'_n} \ldots \mathrm{Bl}_{p'_1}\mathbb{P}^2 & \\
\swarrow^{\pi_n} & & & & \searrow^{\pi'_n} \\
\mathrm{Bl}_{p_{n-1}} \ldots \mathrm{Bl}_{p_1}\mathbb{P}^2 & & & & \mathrm{Bl}_{p'_{n-1}} \ldots \mathrm{Bl}_{p'_1}\mathbb{P}^2 \\
\downarrow \pi_{n-1} & & & & \downarrow \pi'_{n-1} \\
\ldots & & & & \ldots \\
 & \searrow^{\pi_1} & & \swarrow^{\pi'_1} & \\
 & \mathbb{P}^2 & = & \mathbb{P}^2 &
\end{array}$$

An equivalence class of a tuple $(p_1, \ldots, p_n)$ is called a *constellation.* As before, let $X = \mathrm{Bl}_{p_n} \ldots \mathrm{Bl}_{p_1}\mathbb{P}^2$, then the pullback of a line E_0 together with the total transforms of the points p_i - which we denote as before with E_i - give a free generating system of $\mathrm{Pic}(X)$. Also, $E_0^2 = 1$ and $E_i^2 = -1$. Denote by $\widehat{E_i}$ the strict transform in X of the exceptional divisor in $\mathrm{Bl}_{p_i} \ldots \mathrm{Bl}_{p_1}\mathbb{P}^2$ of a point p_i.

Definition 1. (1) *Let* $p_j \neq p_i$ *be points in a constellation. We say that* p_j *is* proximate *to* p_i*, denoted by* $p_j \dashrightarrow p_i$*, if* p_j *is infineteley near to* p_i*.*

(2) *A* cluster *on* $\mathbb{P}^2$ *is a pair* $(\mathcal{K}, \underline{m})$ *consisting of a constellation* $\mathcal{K} = [(p_1, \ldots, p_n)]$ *and an integer vector* $\underline{m} = (m_1, \ldots, m_n)$*. We inductively introduce the* total multiplicities $\widehat{m_i} := m_i + \sum_{p_i \dashrightarrow p_j} m_j$*. The cluster is called nontrivial if* $\mathcal{K} \neq \emptyset$ *and if there is an* i *with* $\widehat{m_i} > 0$*.*

(3) *A cluster* $(\mathcal{K}, \underline{m})$ *satisfies the proximity relations at* p_i *if and only if* $m_i \geq \sum_{p_j \dashrightarrow p_i} m_j$*.*

(4) *Let $(\mathcal{K}, \underline{m})$ be a non empty cluster on $\mathbb{P}^2$ satisfying the proximity relations. Then the ideal corresponding to $(\mathcal{K}, \underline{m})$, $I(\mathcal{K}, \underline{m}) \subset K[x_0, x_1, x_2]$, is defined by*

$$I(\mathcal{K}, \underline{m}) := \{f \in K[x_0, x_1, x_2] \mid \operatorname{mult}_{p_i} C_i \geq \widehat{m_i}\}$$

where $C = \{f = 0\}$ denotes the curve defined by f and C_i denotes the i-th total transform of C, $(\pi_{i-1} \circ \ldots \circ \pi_1)^*(C)$.

Now we can replace our notion of fat point ideals from the introduction: I is called a fat point ideal, if it is the ideal corresponding to a non empty cluster satisfying the proximity relations. If we would allow clusters which do not satisfy the proximity relations, it may happen, that we get the same corresponding ideal for two clusters even though the multiplicities are not equal. For example, if we take the constellation $\mathcal{K} = [(p_1, p_2)]$, where $p_1 \in \mathbb{P}^2$ and $p_2 \in E_1$, then $I(\mathcal{K}, (1, 0)) = I(\mathcal{K}, (1, -1))$. If we restrict to clusters satisfying the proximity relations, the multiplicities are uniquely determined by $I(\mathcal{K}, \underline{m})$. This is an easy consequence of [CA90], Theorem 3.3. This theorem states that every cluster satisfying the proximity relations and where all points are infinitely near to a single point in $\mathbb{P}^2$ can be realized as the cluster that appears if we resolve a singularity of a curve. That is, for such a cluster, we can find a representative $(p_1, \ldots, p_n)$ in the constellation such that the corresponding multiplicities fulfill $m_1 \geq \ldots \geq m_n$. If we have a cluster where not all the points are infinintely near to a single point, we can still find such a representative with the help of [Har98], Lemma 2.6: this lemma allows to find an isomorphism $\Phi : \operatorname{Bl}_{p_n} \ldots \operatorname{Bl}_{p_1} \mathbb{P}^2 \leftarrow \operatorname{Bl}_{p'_n} \ldots \operatorname{Bl}_{p'_1} \mathbb{P}^2$, and a unique permutation σ_Φ of $\{1, \ldots, n\}$ such that $\Phi^*(\widehat{E_i}) = \widehat{E_{\sigma_\Phi(i)}}$, such that $Z = m_1 p_1 + \ldots + m_n p_n$ is equivalent to $Z' = m_{\sigma_\Phi(1)} p'_1 + \ldots + m_{\sigma_\Phi(n)} p'_n$ and $m_{\sigma_\Phi(1)} \geq \ldots \geq m_{\sigma_\Phi(n)}$. We will need this fact in the proof of Theorem B.

Example 2. (1) *Let $p_1 \in \mathbb{P}^2$, $p_2 \in \operatorname{Bl}_{p_1} \mathbb{P}^2 \backslash E_1$, $p_3 \in \operatorname{Bl}_{p_2} \operatorname{Bl}_{p_1 \mathbb{P}^2} \backslash (E_1 \cup E_2)$ and so on. Let $\mathcal{K} = [(p_1, \ldots, p_n)]$ and $\underline{m} = (m_1, \ldots, m_n)$. That $(\mathcal{K}, \underline{m})$ satisfies the proximity relations means nothing else than $m_i \geq 0$ for all i. Then $I(\mathcal{K}, \underline{m}) = \mathfrak{m}_{p_1}^{m_1} \cap \ldots \cap \mathfrak{m}_{p_n}^{m_n}$ where $\mathfrak{m}_{p_i}$ denotes the vanishing ideal of $(\pi_{i-1} \circ \ldots \circ \pi_1)(p_i)$.*

(2) *Let $p_1 = (0 : 0 : 1) \in \mathbb{P}^2$. In $\operatorname{Bl}_{p_1} \mathbb{P}^2$, let p_2 be the point in E_1, corresponding to the line $\{x = 0\}$ in $\mathbb{P}^2$. Let p_3 be the intersection point of $\widehat{E_1}$ and E_2 in $\operatorname{Bl}_{p_2} \operatorname{Bl}_{p_1} \mathbb{P}^2$. Then $([(p_1, p_2, p_3)], (2, 1, 1))$ is a cluster satisfying the proximity relations and the corresponding fat point ideal is $I = \langle x^2 z, xy^2, y^3 \rangle \subset K[x, y, z]$. This is the cluster that appears if we blow up the singularity of the cusp $\{x^2 z - y^3 = 0\}$.*

Remark 3. *An equivalent definition for the proximity relations is the following: A cluster $(\mathcal{K} = [(p_1, \ldots, p_n)], \underline{m})$ satisfies the proximity relations if for all strict transforms $\widehat{E_k}$ of exceptional divisors on the blown up surface X,*

$$(-m_1 E_1 - \ldots - m_n E_n).\widehat{E_k} \geq 0 \text{ for all } k.$$

This formulation is equivalent to the above, as $\widehat{E_k} = E_k - \sum_{p_j \dashrightarrow p_k} E_j$.

See [CA90] for more explanations about clusters.

3. Changing the problem

We want to transform the problem of finding the free resolution of a fat point ideal step by step into a different problem. Let $I = I(\mathcal{K}, \underline{m})$ be a fat point ideal, that is, $(\mathcal{K}, \underline{m})$ is a non empty cluster satisfying the proximity relations. Choose a representative $(p_1, \dots, p_n)$ in $\mathcal{K}$ such that $m_1 \geq \dots \geq m_n$. As before, let $X = \mathrm{Bl}_{p_n} \dots \mathrm{Bl}_{p_1} \mathbb{P}^2$. As $\operatorname{codim} I = 2$ we can use the Auslander-Buchsbaum-Formula to compute the length of the free resolution of I. The free resolution of I has length 1, i.e. it has the following form:

$$0 \to M_1 = \bigoplus_{d \geq 0} R[-d]^{s_d} \to M_0 = \bigoplus_{d \geq 0} R[-d]^{v_d} \to I \to 0.$$

3.1. $v_d = \dim_K \operatorname{coker} \mu_d$ **and** $s_d = v_d - h_I(d) + 3h_I(d-1) - 3h_I(d-2) + h_I(d-3)$**.** v_d is the number of generators in degree d in a minimal generation system of I. Denote by μ_d the *multiplication map*

$$\mu_d : I_{d-1} \otimes R_1 \longrightarrow I_d.$$

Then $v_d = \dim_K \operatorname{coker} \mu_d$, as the image of μ_d contains all those forms of degree d in I that are a multiple of some form of lower degree in I. The equation for the s_d, $s_d = v_d - h_I(d) + 3h_I(d-1) - 3h_I(d-2) + h_I(d-3)$, can be seen if we tensor the free resolution with K and consider the Koszul complex of K.

3.2. $\dim \operatorname{coker} \mu_d = \dim_K \operatorname{coker} \mu_{X,F,E_0}$ **for a special linear system** F **on the blown up surface** X **and** $h_I(d) = h^0(X, \mathcal{O}_X(F))$**.** As already mentioned in the introduction, $h_I(d) = h^0(X, \mathcal{O}_X(dE_0 - m_1E_1 - \dots - m_nE_n))$. This follows from the fact that $I = \pi_*(\mathcal{O}_X(-m_1E_1 - \dots - m_nE_n)$ (where $\pi : X \to \mathbb{P}^2$ is the map of all blow ups) which can be seen considering local coordinates. Let F_d denote the linear system $F_d = dE_0 - m_1E_1 - \dots - m_nE_n$ on X. Then $h_I(d) = h^0(X, \mathcal{O}_X(F_d))$. Also, the multiplication map $\mu_d : I_{d-1} \otimes R_1 \longrightarrow I_d$ corresponds to a map

$$\mu_{X,F_d,E_0} : H^0(X, \mathcal{O}_X(F_{d-1})) \otimes H^0(X, \mathcal{O}_X(E_0)) \longrightarrow H^0(X, \mathcal{O}_X(F_d)).$$

3.3. $h^0(X, \mathcal{O}_X(F)) = h^0(X, \mathcal{O}_X(H))$ **if** $F = H + N$ **where** F **is fixed and** H **is nef and** $\dim_K \operatorname{coker} \mu_{X,F,E_0} = \dim_K \operatorname{coker} \mu_{X,H,E_0} + h^0(X, \mathcal{O}_X(F+E_0)) - h^0(X, \mathcal{O}_X(H+E_0))$**.** If we have a decomposition of F such that $F = H + N$ where N is fixed in F and H is nef, then $h^0(X, \mathcal{O}_X(F)) = h^0(X, \mathcal{O}_X(H))$. Furthermore, by the identification $H^0(X, \mathcal{O}_X(H)) = H^0(X, \mathcal{O}_X(F))$ the image of

$$H^0(X, \mathcal{O}_X(H)) \otimes H^0(X, \mathcal{O}_X(E_0)) \longrightarrow H^0(X, \mathcal{O}_X(H + E_0)) \longrightarrow H^0(X, \mathcal{O}_X(F + E_0))$$

is equal to the image of μ_{X,F,E_0}, and hence we get the relation:

$$\dim_K \operatorname{coker} \mu_{X,F,E_0} = \dim_K \operatorname{coker} \mu_{X,H,E_0} + h^0(X, \mathcal{O}_X(F + E_0)) - h^0(X, \mathcal{O}_X(H + E_0)).$$

So finally, if we can find

(1) a decomposition in nef part H and fixed part N for a given linear system on X,
(2) $h^0(X, \mathcal{O}_X(H))$, and
(3) $\dim_K \operatorname{coker} \mu_{X,H,E_0}$ for a nef linear system H,

we can give the free resolution of the ideal I. However, it is not possible to solve these problems in general. In the present paper we investigate them in the situation where the points lie on a conic. The solution will be presented in the following section. For a more detailed account on the facts mentioned in this section, see [Mar03], Chapters 1 and 2.

4. Points on a conic

Let $I = I(\mathcal{K}, \underline{m})$ be a fat point ideal, and let Q' be a conic in $\mathbb{P}^2$. Let $(p_1, \ldots, p_n) \in \mathcal{K}$ such that $m_1 \geq \ldots \geq m_n$ and assume that for all i, p_i lies on the strict transform of Q' in $\mathrm{Bl}_{p_{i-1}} \ldots \mathrm{Bl}_{p_1} \mathbb{P}^2$. Denote by Q the stric transform of Q' in X. Then the linear system of Q is $2E_0 - E_1 - \ldots - E_n$ and so the latter is non empty. Furthermore, the linear system of $E_0 + Q = -K_X$ is non empty. These two non empty linear systems are important to get solutions for our three problems.

4.1. **The decomposition.** The first problem is to find a decomposition of a linear system F in nef part and fixed part. Assume that $F \neq \emptyset$. The idea is, if we have a finite number of irreducible curves with negative self-intersection on X, we can just intersect F with those curves and subtract those which have negative intersection with F. The remaining system is nef.

In the case that the points lie on a conic, the number of curves of negative self-intersection is finite. More precisely, the only curves of negative self-intersection are components of E_i for some $i > 0$, components of Q or strict transforms of lines through the points. The idea to prove this is to assume that a curve C of negative self-intersection would fulfill $C.E_0 > 2$. Then $-C.K_X = C.(Q + E_0) \geq C.E_0 > 2$ as C cannot be a component of Q, and we get a contradiction to negative self-intersection by the adjunction formula, as $p_a(C) \geq 0$. For a detailed proof, see [Har98], Lemma III.i.1.(a).

4.2. $h^0(X, \mathcal{O}_X(H))$ **for a nef linear system** H**.** The second problem is to compute $h^0(X, \mathcal{O}_X(H))$ for a nef linear system H on X. [Har97], Lemma II.2(c), tells us that $h^2(X, \mathcal{O}_X(H)) = 0$ for a nef linear system H on any surface X with $-K_X \neq \emptyset$, and [Har98], Lemma III.I.1 (b), tells us that $h^1(X, \mathcal{O}_X(H)) = 0$ for a nef linear system H on X, where the blown up points lie on a conic. The same lemma tells us that H has no fixed components, that is if we have a decomposition of a linear system F in part N which is fixed in F and a part H which is nef, then N is the fixed part of F and H has no fixed part. So by Riemann-Roch we get

$$h^0(X, \mathcal{O}_X(H)) = (H^2 - K_X.H)/2 + 1.$$

4.3. $\dim_K \operatorname{coker} \mu_{X,H,E_0}$ **for a nef linear system** H**.** The third problem is to compute $\dim_K \operatorname{coker} \mu_{X,H,E_0}$ for a nef linear system H on X. Harbourne gives a solution for this problem, too, but as one Lemma ([Har98], Lemma II.5) which he uses in his proof seems not to hold in the generality stated there, we will give a correct version of his proof here. We need a lemma in advance:

Lemma 4. *Let N be a reduced divisor on X such that $|N| \neq \emptyset$ and $h^0(X, \mathcal{O}_X(N + K_X)) = 0$. Assume N has at most two components. Let F and G be divisors on X meeting every component of N nonnegatively. If N has only one component, then the multiplication map $\mu_{N,F,G} : H^0(N, \mathcal{O}_N(F)) \otimes H^0(N, \mathcal{O}_N(G)) \to H^0(N, \mathcal{O}_N(F + G))$ fulfills* $\operatorname{coker} \mu_{N,F,G} = 0$*. If N has two components, then we get the same result if we have $G.C_2 \geq C_1.C_2$, $G.C_1 \geq C_2.C_1$ and $F.C_1 \geq C_2.C_1$, where C_1 and C_2 are the two components of N.*

Proof. As $0 = H^0(X, \mathcal{O}_X(N + K_X)) = H^2(X, \mathcal{O}_X(-N))$ by assumption, we see from the cohomology sequence of the structure sequence of N in X that $h^1(N, \mathcal{O}_N) = 0$. Then by [Art62], Theorem 1.7, it follows that the components of N are rational.

Assume first that $N \cong \mathbb{P}^1$ has only one component. As F and G meet N non negatively, we get effective divisors on N and the map $\mu_{N,F,G} : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(F)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(G)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(F+G))$ maps a tensor product of two polynomials of degree $F.N$ respectively $G.N$ in two variables to their product. This map is surjective, as $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(F+G))$ is generated by the monomials in degree $F.N + G.N$ and we can give a preimage for every such monomial easily. So $\mu_{N,F,G}$ is surjective and $\dim_K \operatorname{coker} \mu_{N,F,G} = 0$.

Now assume that N has two components C_1 and C_2. By the same argument as above, we see $\operatorname{coker} \mu_{C_2,F,G} = 0$ and $\operatorname{coker} \mu_{C_1,F-C_2,G} = 0$ since by assumption $(F - C_2).C_1 \geq 0$. Consider the sequence

$$0 \to \mathcal{O}_{C_1}(-C_2) \to \mathcal{O}_N \to \mathcal{O}_{C_2} \to 0.$$

From this we can get the following diagram, after tensoring the sequence with F, respectively the corresponding long cohomology sequence with $H^0(N, \mathcal{O}_N(G))$ (in the diagram, we use $H^0(N,G)$ as a shortcut for $H^0(N, \mathcal{O}_N(G))$ and analogous shortcuts):

$$\begin{array}{ccccccccc} 0 & \to & H^0(C_1, F - C_2) \otimes H^0(N,G) & \to & H^0(N,F) \otimes H^0(N,G) & \to & H^0(C_2,F) \otimes H^0(N,G) & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \mu_{N,F,G}} & & \downarrow & & \\ 0 & \longrightarrow & H^0(C_1, F+G-C_2) & \longrightarrow & H^0(N, F+G) & \longrightarrow & H^0(C_2, F+G) & \longrightarrow & 0 \end{array}$$

The rows are exact, because $h^1(C_1, F - C_2) = h^1(C_1, F + G - C_2) = 0$ (by Serre duality, and since C_1 is rational and as $(F - C_2).C_1 \geq 0$ by assumption) and since $H^0(N,G)$ is free. From this diagram we get with the Snake Lemma an exact sequence

$$\operatorname{coker}\big(H^0(C_1, F - C_2) \otimes H^0(N,G) \to H^0(C_1, F + G - C_2)\big) \longrightarrow \operatorname{coker} \mu_{N,F,G} \longrightarrow$$

$$\operatorname{coker}\big(H^0(C_2, F) \otimes H^0(N,G) \to H^0(C_2, (F+G))\big) \longrightarrow 0\ .$$

As $h^1(C_1, \mathcal{O}_{C_1}(G - C_2)) = 0$ (again by Serre duality, and since C_1 is rational and $(G - C_2).C_1 \geq 0$) we can see from the sequence above that $H^0(N, \mathcal{O}_N(G)) \to H^0(C_2, \mathcal{O}_{C_2}(G))$ is surjective and so

$$\operatorname{coker}\big(H^0(C_2, \mathcal{O}_{C_2}(F)) \otimes H^0(N, \mathcal{O}_N(G)) \to H^0(C_2, \mathcal{O}_{C_2}(F+G))\big) = \operatorname{coker} \mu_{C_2,F,G} = 0.$$

Reversing the role of C_1 and C_2 in the sequence we also get $H^0(N, \mathcal{O}_N(G)) \to H^0(C_1, \mathcal{O}_{C_1}(G))$ is surjective (as $h^1(C_2, \mathcal{O}_{C_2}(G - C_1)) = 0$ as $(G - C_1).C_2 \geq 0$) and hence

$$\operatorname{coker}\big(H^0(C_1, \mathcal{O}_{C_1}(F - C_2)) \otimes H^0(N, \mathcal{O}_N(G)) \to H^0(C_1, \mathcal{O}_{C_1}(F + G - C_2))\big) = \operatorname{coker} \mu_{C_1,F-C_2,G} = 0.$$

But then of course $\operatorname{coker} \mu_{N,F,G} = 0$. □

Lemma 5. *Let $X = Bl_{p_n} \ldots Bl_{p_1} \mathbb{P}^2$ such that $p_1, \ldots, p_n$ lie on a conic. Let H be a nef linear system on X. Then $\dim_K \operatorname{coker} \mu_{X,H,E_0} = 0$.*

Proof. We know that we can write $H = a_0E_0 - a_1E_1 - \ldots - a_nE_n$. H nef implies $a_i \geq 0$ for all i. The proof will be an induction on a_0. Assume $a_0 = 0$. Then $a_i = 0$ for all i as otherwise $H = \emptyset$ which is not possible by [Har98], Lemma 3.1.1.(b). Assume $a_0 = 1$. Then only one a_i can be non zero, as otherwise H would not be nef. So without restriction $H = E_0 - E_1$. Consider the sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(H) \to \mathcal{O}_H(H) \to 0.$$

Taking the long cohomology sequence and tensoring it with $H^0(X, \mathcal{O}_X(E_0))$, we can again see with the Snake Lemma that $\operatorname{coker} \mu_{X,H,E_0} = \operatorname{coker} \mu_{H,H,E_0}$. But the latter is zero by Lemma 4, as H has only one irreducible component, H and E_0 intersect this component non negatively and $H^0(X, \mathcal{O}_X(H + K_X)) = 0$.

So assume $a_0 \geq 2$. Then we can subtract the linear system of the conic Q (which we will by abuse of notation also call Q) from H, and we still get a nef linear system, which can be tested intersecting $H-Q$ with the possible curves of negative self-intersection (see [Har98], Lemma III.i.1.(c)). So the induction assumption tells us that $\dim_K \operatorname{coker} \mu_{X,H-Q,E_0} = 0$. With Ramanujan-vanishing (see [Ram74], Theorem 1) we can see that $h^1(X, \mathcal{O}_X(H - Q)) = 0$ and $h^1(X, \mathcal{O}_X(E_0 - Q)) = 0$. The first gives us an exact sequence

$$0 \to H^0(X, \mathcal{O}_X(H - Q)) \to H^0(X, \mathcal{O}_X(H)) \to H^0(Q, \mathcal{O}_Q(H)) \to 0$$

which we can tensor with $H^0(X, E_0)$ and apply Snake Lemma to get an exact cokernel sequence

$$\operatorname{coker} \mu_{X,H-Q,E_0} \to \operatorname{coker} \mu_{X,H,E_0} \to \\ \operatorname{coker} \big(H^0(Q, \mathcal{O}_Q(H)) \otimes H^0(X, \mathcal{O}_X(E_0)) \to H^0(Q, \mathcal{O}_Q(H + E_0)) \big).$$

The latter tells us that $H^0(X, \mathcal{O}_X(E_0)) \to H^0(Q, \mathcal{O}_Q(E_0))$ is surjective, and so the last cokernel in the sequence is just $\operatorname{coker} \mu_{Q,H,E_0}$.

We want to see that $\dim_K \operatorname{coker} \mu_{Q,H,E_0}$ is also zero, as then $\operatorname{coker} \mu_{X,H,E_0} = 0$ with the induction assumption. To see this, apply Lemma 4 with $N = Q$, $F = H$ and $G = E_0$. As $Q = 2E_0 - E_1 - \ldots - E_n$, $h^0(X, \mathcal{O}_X(Q + K_X)) = 0$. There are two cases: either Q has only one component, or Q is a union of two lines C_1 and C_2. These lines can intersect at most with multiplicity 1, so $H.C_1 \geq C_1.C_2 = 1$ and $1 = E_0.C_2 = E_0.C_1 = C_1.C_2$. In any case, the assumptions of Lemma 4 are fulfilled and we get $\operatorname{coker} \mu_{Q,H,E_0} = 0$ and hence with the cokernel sequence from above, $\operatorname{coker} \mu_{X,H,E_0} = 0$. □

5. The proof of theorem B

With these results, we are now able to compute a free resolution for points that lie on a conic. The Theorems A and B give a free resolution explicitly for two cases: if all but one of the points lie on a line and the point which is not on the line has multiplicity 1, and if all points lie on a smooth conic and the highest multiplicity occurs four times. The proofs for both theorems are quite similar, using the results from above. Here, we just want to give a proof for Theorem B.

General Requirement 6. *Let $I = I(\mathcal{K}, \underline{v})$ be a fat point ideal, that is, $(\mathcal{K}, \underline{v})$ is a nonempty cluster satisfying the proximity relations. Assume that the highest multiplicity, m, i.e. the highest entry in the vector $\underline{v}$, occurs four times in $\underline{v}$. Choose a representative $(p, q, r, s, p_1, \ldots, p_n)$ in $\mathcal{K}$ such that the highest multiplicity m is associated to p, q, r and s and such that for the other associated multiplicities, $m_1 \geq \ldots \geq m_n$ holds. Assume that p, q, r, s and $p_1, \ldots, p_n$ lie on a conic, respectively on the strict transform of the conic. Let $X = \mathrm{Bl}_{p_n} \ldots \mathrm{Bl}_{p_1} \mathrm{Bl}_s \mathrm{Bl}_r \mathrm{Bl}_q \mathrm{Bl}_p \mathbb{P}^2$ and let Q denote the strict transform of the conic in X.*

The restriction that the multiplicity m occurs four times is necessary, as otherwise the computations would be too complicated. We want to calculate the free resolution of the ideal I. First we have to compute the smallest degree d such that $h_I(d) > 0$ - the corresponding linear systems are then non empty and we can compute their decomposition as described in 4.1. As the linear system of all conics passing through the four points p, q, r and s has dimension one, we can certainly find a conic $Q' \subset X$ different from Q that goes through the four points p, q, r and s. If these four points are all of level zero (i.e. they are not infinitely near to any other point), $\pi(Q')$ is a conic through the four points in $\mathbb{P}^2$. Then $\pi(Q')$ has multiplicity one in each of the four points. If some of the four points are infinitely near, $\pi(Q')$ goes through their images in $\mathbb{P}^2$, and the sum of the multiplicities in the image points is in any case bigger than or equal to four. Assume that $h_I(d) > 0$. Then with Bézout's Theorem (see [Har77], page 53) we know that

$$2d = \deg\big(\pi(F_d)\big) \cdot deg\big(\pi(Q')\big) \geq 4m$$

so $d \geq 2m$. But as $m \cdot Q \in I_{2m}$ we can see that $h_I(2m) > 0$ and so the smallest degree for which $h_I(d) > 0$ is $2m$. So

$$F_{2m} = 2mE_0 - mE_p - mE_q - mE_r - mE_s - m_1E_1 - \ldots - m_nE_n$$

is in a non empty linear system and we have to find curves in the fixed part of this linear system in order to calculate $h^0(X, \mathcal{O}_X(F_{2m}))$. From 4.1 we know that the only irreducible curves that we can find in the fixed part, are components of E_i for some $i \in \{p, q, r, s, 1 \ldots, n\}$, $E_0 - E_i - E_j$ where $i \neq j$ and $i, j \in \{p, q, r, s, 1, \ldots, n\}$, or the conic Q itself, which is irreducible.

Let us try Q first.

$$F_{2m}.Q = 4m - m - m - m - m - m_1 - \ldots - m_n < 0$$

so Q is fixed, and after subtracting it, we get

$$\begin{aligned} F_{2m} - Q = &(2m-2)E_0 - (m-1)E_p - (m-1)E_q - (m-1)E_r - (m-1)E_s \\ &- (m_1 - 1)E_1 - \ldots - (m_n - 1)E_n. \end{aligned}$$

Intersecting this new linear system again with Q, we get

$$4m - 4 - 4(m-1) - \sum_{i=1}^{n}(m_i - 1)$$

which is negative if $m_1 \geq 2$. If this is the case, we can again subtract Q and we can see that going on like this, the first 4 points will "cancel" the positive part in the sum coming from $(F_{2m} - iQ).E_0 \cdot 2$, and the sum of the remaining n points counts negatively.

So we can subtract Q altogether m_n times. But again, $(F_{2m} - m_n \cdot Q).Q < 0$, and we can subtract Q one more time, ending up with a linear system F'_{2m} such that $F'_{2m}.E_n = -1$. But then we can subtract E_n and have again a linear system with only negative multiplicities for the E_i, and because then the sum of the multiplicities of the $n-1$ remaining points counts purely negatively again, we can as above subtract Q once more and so on.

Finally, we end up with the linear system

$$H_{2m} = (2m - 2m_1)E_0 - (m - m_1)E_p - (m - m_1)E_q - (m - m_1)E_r - (m - m_1)E_s.$$

Now trying all irreducible curves of negative self-intersection, and seeing that they intersect H_{2m} non negatively, we find that H_{2m} is nef. Remark 3 tells us that we do not have to check the components of the E_i. But then we know by 4.2 that $h^1(X, \mathcal{O}_X(H_{2m})) = 0$ and

$$h^0(X, \mathcal{O}_X(H_{2m})) = \big((2m-2m_1+1)(2m-2m_1+2)-4(m-m_1)(m-m_1+1)\big)/2 = m-m_1+1.$$

Next we have to ask ourselves up to which degree d the curve $m_1 \cdot Q$ is fixed in F_d. Call $r_{m_i} = \sharp\{j \in \{1, \dots, n\} \mid m_j = m_i\}$. If $r_{m_1} = 1$, then already $(F_{2m+1} - (m_1 - 2) \cdot Q).Q = 4m + 2 - 4m_1 + 8 - 4m + 4m_1 - 8 - 2 = 0$, so Q is not for sure fixed the $(m_1 - 1)-$st time in F_{2m+1}, as this intersection is not less than zero, and so only $(m_1 - 2) \cdot Q$ is in the fixed part of F_{2m+1}.

Notation 7. *We use the following notation*

$$\begin{aligned} F_d - t \cdot Q = (d-t)E_0 - \max(m-t, 0)E_p - \ldots - \max(m-t, 0)E_s \\ - \max(m_1 - t, 0)E_1 - \ldots - \max(m_n - t)E_n. \end{aligned}$$

The change of the fixed part described above happens because the four points with multiplicity m always "cancel" the "$2m-$part" of $F_{2m+a}.E_0$ - so if we enlarge the degree by one, we get two more in the E_0-part of the intersection $F_{2m+a}.Q$, and two less, each time we subtract Q, but we get only one less in the $E_1, \dots, E_n-$part when we subtract Q for the last times, because then only E_1 is left. So for the first degrees, up to the degree where E_2 appears with a multiplicity bigger than zero in the fixed part of F_d, there is a difference in the change of the fixed parts: for every degree d F_d has Q two times less in its fixed part than F_{d-1}. Call $c := m_1 - m_2$. As long as $a \leq \lfloor c/2 \rfloor$, the fixed part of F_{2m+a} is $(m_1 - 2a) \cdot Q$, as

$$\begin{aligned} (F_{2m+a}-&(m_1 - 2a - 1)Q).Q \\ =&((2m + a - 2(m_1 - 2a - 1))E_0 - (m - m_1 + 2a + 1)E_p \\ &- \ldots - (m - m_1 + 2a + 1)E_s - (2a + 1)E_1).Q \\ =&4m + 2a - 4m_1 + 8a + 4 - 4m + 4m_1 - 8a - 4 - 2a - 1 = -1 < 0 \end{aligned}$$

and

$$\begin{aligned} (F_{2m+a}-&(m_1 - 2a)Q).Q \\ =&((2m + a - 2(m_1 - 2a))E_0 - (m - m_1 + 2a)E_p \\ &- \ldots - (m - m_1 + 2a)E_s - (2a)E_1).Q \\ =&4m + 2a - 4m_1 + 8a - 4m + 4m_1 - 8a - 2a = 0. \end{aligned}$$

The next step now depends on whether c is even or odd, because in the first case, $(m_1-c)Q$ is fixed in $F_{2m+c/2}$, and in the other case, $(m_1 - c + 1)Q$ is fixed in $F_{2m+(c-1)/2}$, and we have to decide about what is fixed in the following degrees with this knowledge. So we make a distinction on whether c is even or odd.

Let us start with the case that c is even. As we have already seen, we get two times Q less in the fixed part for every degree from $2m$ up to $2m + c/2$. So for $0 \leq a \leq c/2$, the fixed part of F_{2m+a} is $(m_1 - 2a)Q$ and we can compute

$$\begin{aligned} h_I(2m+a) =& h^0\big(X, \mathcal{O}_X(F_{2m+a})\big) = h^0\big(X, \mathcal{O}_X(F_{2m+a} - (m_1 - 2a)Q)\big) \\ =& h^0\big(X, \mathcal{O}_X((2m+a-2(m_1-2a))E_0 - (m-m_1+2a)E_p \\ & - \ldots - (m-m_1+2a)E_s - 2aE_1)\big) \\ =& \frac{1}{2}((2m+5a-2m_1+1)(2m+5a-2m_1+2) \\ & - 4(m-m_1+2a)(m-m_1+2a+1) - 2a(2a+1)). \end{aligned}$$

Now for the degrees following $2m + c/2$, it depends on how often the multiplicity m_2 appears, up to which degree $(m_1 - c)Q$ is in the fixed part. From the degree where Q is less than m_2 times in the fixed part, we have more than only one E_i left in the nef part of the linear system F_{2m+a}, and so in the intersection sum $H_{2m+a}.Q$, we get two less in the E_0- part when we subtract Q and two or more less in the $E_1, \ldots, E_n-$part, so it can no longer happen that the fixed components differ by $2Q$ for each degree. This means we can go on in the following way: we know that for the degrees following $2m + c/2$, m_2Q is in the fixed part, and we have to ask ourselves which is the degree for which that is no longer true, i.e. for which degree $2m+a$ only $(m_2-1)Q$ is in the fixed part. The inequality that tells us Q is fixed the m_2-th time is $(F_{2m+a} - (m_2 - 1)Q).Q < 0$, so the degree for which this is no longer true, is the degree for which

$$\begin{aligned} 0 &= (F_{2m+a} - (m_2-1)Q).Q \\ &= ((2m+a-2m_2+2)E_0 - (m-m_2+1)E_p \\ &- \ldots - (m-m_2+1)E_s - (c+1)E_1 - E_2 - \ldots - E_{\mu_{m_2}+1}).Q \\ &= 4m+2a-4m_2+4-4m+4m_2-4-c-1-\mu_{m_2} = 2a-c-1-\mu_{m_2} \end{aligned}$$

$$\Leftrightarrow 2a = c + 1 + \mu_{m_2} \tag{5.1}$$

where μ_{m_2} is defined with the help of the *Young diagram* of $(m_2, \ldots, m_n)$, this is a diagram consisting of $n-1$ rows of boxes, the bottom row consisting of m_2 boxes, the second of m_3 boxes, and so on, and the top one consisting of m_n boxes. The *conjugate Young diagram* is the diagram $(\mu_1, \ldots, \mu_{m_2})$ where μ_i is the number of boxes in the i-th column of the Young diagram of $(m_2, \ldots, m_n)$.

But the equation above might not be solvable as $c + 1 + \mu_{m_2}$ might be odd. Then we have to ask ourselves, for which value of a the intersection of $F_{2m+a} - (m_2 - 1)Q$ with Q is no longer negative. $2a$ is of course always even, and if $c + 1 + \mu_{m_2}$ is odd, then $(F_{2m+a} - (m_2 - 1)Q).Q = 2a - (c+1+\mu_{m_2})$ is odd. So it can never be zero if we enlarge a by steps of one, but it will be -1 and then 1 and the next step - and this is the step to stop. To sum up, F_{2m+a} has no longer m_2Q in its fixed component if $a = \lceil \frac{c+1+\mu_{m_2}}{2} \rceil$. So we know for all

$$c/2 \leq a < \lceil \frac{c+1+\mu_{m_2}}{2} \rceil, \text{ that}$$

$$h_I(2m+a) = h^0(X, \mathcal{O}_X(F_{2m+a})) = h^0(X, \mathcal{O}_X(F_{2m+a} - m_2Q))$$

and as $h^1(X, \mathcal{O}_X(F_{2m+a} - m_2Q)) = 0$ since this linear system is nef, the latter can be calculated easily with Riemann-Roch.

Definition 8. *We will denote a degree d such that F_{d-1} has $i \cdot Q$ as fixed component whereas F_d only has $(i-1) \cdot Q$ as fixed component a* special degree *of the ideal I.*

We now have to ask, up to which degree $(m_2-1)Q$ is in the fixed part, that is, we have to find special degrees. Let us enumerate not the special degrees (as they depend on whether $c+1+\mu_{m_2}$ or a similar sum is even or odd), but these sums that tell us how big $2a$ (as in the equation 5.1) may be - call $a_{m_2}=c+1+\mu_{m_2}$, the first special degree is then $2m+\lceil a_{m_2}/2\rceil$. We find a_{m_2-i} if we ask us for which degree $2m+a$, $(m_2-i)Q$ is no longer in the fixed part, i.e. for which degree $(F_{2m+a}-(m_2-i-1)Q).Q$ is no longer negative.

$$\begin{aligned}(F_{2m+a}-&(m_2-i-1)Q).Q\\ =&((2m+a-2(m_2-i-1))E_0-(m-m_2+i+1)E_p\\ &-\ldots-(m-m_2+i+1)E_s-(c+i+1)E_1-(i+1)E_2\\ &-\ldots-(i+1)E_{\mu_{m_2}+1}-iE_{\mu_{m_2}+2}\\ &-\ldots-iE_{\mu_{m_2-1}+1}-\ldots-E_{\mu_{m_2-i-1}+2}-\ldots-E_{\mu_{m_2-i}+1}).Q\\ =&2a-(c+i+1+\mu_{m_2}+\ldots+\mu_{m_2-i})\end{aligned}$$

and so $a_{m_2-i}=c+i+1+\mu_{m_2}+\ldots+\mu_{m_2-i}$.

So we know that for each

$$\lceil\frac{a_{m_2-i+1}}{2}\rceil\leq a<\lceil\frac{a_{m_2-i}}{2}\rceil,$$

$$h_I(2m+a)=h^0(X,\mathcal{O}_X(F_{2m+a}))=h^0(X,\mathcal{O}_X(F_{2m+a}-(m_2-i)Q))$$

where $a_{m_2+1}:=c$ for completeness.

With these results, we are able to compute the Hilbert function for each degree in the case that c is even.

The case that c is odd is done analogously, the result is:

(1) for $0\leq a\leq\frac{c-1}{2}$

$$h_I(2m+a)=h^0(X,\mathcal{O}_X(F_{2m+a}))=h^0(X,\mathcal{O}_X(F_{2m+a}-(m_1-2a)Q));$$

(2) for $\frac{c+1}{2}\leq a<\lceil\frac{a_{m_2}}{2}\rceil$

$$h_I(2m+a)=h^0(X,\mathcal{O}_X(F_{2m+a}-m_2Q));$$

(3) for $\lceil\frac{a_{m_2-i}}{2}\rceil\leq a<\lceil\frac{a_{m_2-i-1}}{2}\rceil$

$$h_I(2m+a)=h^0(X,\mathcal{O}_X(F_{2m+a}-(m_2-i-1)Q)).$$

Now we have to compute the v_d, the graded Betti numbers of the first module of the free resolution.

Lemma 9. *Under the given assumptions, $v_d=0$ for all degrees d which are not special and not equal to $m+1$.*

Proof. From 3.3 and since $\dim\operatorname{coker}\mu_{X,H,E_0}=0$ for the nef part H of the linear system F_{d-1} by 4.3, we know that

$$v_d=\dim\operatorname{coker}\mu_{X,F_{d-1},E_0}=h^0(X,\mathcal{O}_X(F_{d-1}+E_0))-h^0(X,\mathcal{O}_X(H+E_0)).$$

But we compute $h^0(X,\mathcal{O}_X(F_{d-1}+E_0))=h^0(X,\mathcal{O}_X(F_d))$ as above: we find the fixed part of F_d and compute h^0 of the remaining nef part. As d is not a special degree, we know both F_{d-1} and F_d have the same fixed part $i\cdot Q$. But then

$$\begin{aligned}v_d&=h^0(X,\mathcal{O}_X(F_d))-h^0(X,\mathcal{O}_X((F_{d-1}-i\cdot Q)+E_0))\\ &=h^0(X,\mathcal{O}_X(F_d-i\cdot Q))-h^0(X,\mathcal{O}_X(F_d-i\cdot Q))=0.\end{aligned}$$

□

So we only have to consider the cases where the fixed parts differ. This is for example the case for the degrees less than $c/2$, whether c is even or not, we know that for $a \leq c/2$ respectively $a \leq (c-1)/2$, F_{2m+a} has $(m_1-2a)Q$ as fixed part. So for $a \leq c/2$ respectively $a \leq (c-1)/2$,

$$v_{2m+a} = h^0(X, \mathcal{O}_X(F_{2m+a})) - h^0(X, \mathcal{O}_X(H_{2m+a-1} + E_0))$$

where H_{2m+a-1} is the nef part of F_{2m+a-1}

$$\begin{aligned}
=&h^0(F_{2m+a} - (m_1 - 2a)Q)) - h^0(F_{2m+a-1} - (m_1 - 2(a-1))Q + E_0) \\
=&h^0((2m + 5a - 2m_1)E_0 - (m - m_1 + 2a)E_p - \ldots - (m - m_1 + 2a)E_s - 2aE_1) \\
&- h^0((2m + 5a - 2m_1 - 4)E_0 - (m - m_1 + 2a - 2)E_p \\
&- \ldots - (m - m_1 + 2a - 2)E_s - (2a - 2)E_1) = 3
\end{aligned}$$

where we write $h^0(F_d)$ as a shortcut for $h^0(X, \mathcal{O}_X(F_d))$. In the case that c is odd, we know that also the fixed parts of $F_{2m+(c-1)/2}$ and $F_{2m+(c+1)/2}$ differ, so we have to compute $v_{2m+(c+1)/2}$. An analogous computation as above shows that $v_{2m+(c+1)/2} = 2$. At last, we have to compute v_d for the special degrees:

Lemma 10. *For all special degrees $d = 2m + \lceil \frac{a_{m_2-i}}{2} \rceil$, $v_d = 1$ if a_{m_2-i} is even, and $v_d = 2$ if a_{m_2-i} is odd.*

Proof. Let $d = 2m + \lceil \frac{a_{m_2-i}}{2} \rceil$. Then F_d has the fixed part $(m_2 - i - 1) \cdot Q$ whereas F_{d-1} has the fixed part $(m_2 - i) \cdot Q$. So

$$\begin{aligned}
h^0(F_d)-&h^0(H_{d-1} + E_0) \\
&=h^0(F_d - (m_2 - i - 1) \cdot Q) - h^0((F_{d-1} - (m_2 - i) \cdot Q) + E_0) \\
&=h^0((2m + \lceil a_{m_2-i}/2 \rceil - 2m_2 + 2i + 2)E_0 - (m - m_2 + i + 1)E_p - \ldots \\
&\quad - (m - m_2 + i + 1)E_s - (c + i + 1)E_1 - (i + 1)E_2 - \ldots \\
&\quad - (i + 1)E_{\mu_{m_2}+1} - \ldots - E_{\mu_{m_2-i}+1}) \\
&\quad - h^0((2m + \lceil a_{m_2-i}/2 \rceil - 2m_2 + 2i)E_0 - (m - m_2 + i)E_p - \ldots \\
&\quad - (m - m_2 + i)E_s - (c + i)E_1 - iE_2 - \ldots - iE_{\mu_{m_2}+1} - \ldots - E_{\mu_{m_2-i-1}+1}) \\
&=\frac{1}{2}((2m + \lceil a_{m_2-i}/2 \rceil - 2m_2 + 2i + 3)(2m + \lceil a_{m_2-i}/2 \rceil - 2m_2 + 2i + 4) \\
&\quad - 4(m - m_2 + i + 1)(m - m_2 + i + 2) - (c + i + 1)(c + i + 2) \\
&\quad - \mu_{m_2}(i + 1)(i + 2) - (\mu_{m_2-1} - \mu_{m_2})i(i + 1) - \ldots - (\mu_{m_2-i} - \mu_{m_2-i-1}) \cdot 1 \cdot 2) \\
&\quad - \frac{1}{2}(2m + \lceil a_{m_2-i}/2 \rceil - 2m_2 + 2i + 1)(2m + \lceil a_{m_2-i}/2 \rceil - 2m_2 + 2i + 2) \\
&\quad - 4(m - m_2 + i)(m - m_2 + i + 1) - (c + i)(c + i + 1) - \mu_{m_2} \cdot i \cdot (i + 1) \\
&\quad - \ldots - (\mu_{m_2-i-1} - \mu_{m_2-i-1}) \cdot 1 \cdot 2) \\
&=2 \cdot \lceil a_{m_2-i}/2 \rceil + 1 - c - i - 1 - (i + 1)\mu_{m_2} - i \cdot \sharp\{i | m_i = m_2 - 1\} \\
&\quad - \ldots - \sharp\{i | m_i = m_2 - i\} \\
&=2 \cdot \lceil a_{m_2-i}/2 \rceil + 1 - c - i - 1 - \mu_{m_2} - \ldots - \mu_{m_2-i} \\
&=2 \cdot \lceil a_{m_2-i}/2 \rceil + 1 - a_{m_2-i}
\end{aligned}$$

and the latter is 1 if a_{m_2-i} is even and 2 if a_{m_2-i} is odd. □

For the first degree for which $h_I(d)$ is nonzero, we have $v_{2m} = h_I(2m) = m - m_1 + 1$.

The calculation of the s_d, the Betti numbers of the second module in the free resolution, turns out to be much more complicated. As it can be calculated easily, $s_d = 0$ as long as the linear systems F_d, F_{d-1}, F_{d-2} and F_{d-3} have the same fixed part and $v_d = 0$. The cases for which $v_d \neq 0$ are easy to check. But the cases where the fixed parts of the linear systems vary are much more numerous, because for the degrees less than $c/2$ respectively $(c+1)/2$, if c is odd, the fixed part changes from degree to degree by $2Q$, and for bigger d, the fixed part changes at the special degrees. Here, we just want to give one example for a calculation of the s_d, for a complete investigation, see [Mar03]. So let d be a special degree, $d = 2m + \lceil \frac{a_{m_2-i}}{2} \rceil$ for some i. Then v_d depends on whether a_{m_2-i} is even or odd, but we will just insert v_d in the calculation and insert the correct value for the two cases later. Assume $H_{2m+\lceil \frac{a_{m_2-i}}{2} \rceil} = d'E_0 - m'E_p - \ldots - m'E_s - m_1'E_1 - \ldots - m_n'E_n$, then $d' = 2m + \lceil \frac{a_{m_2-i}}{2} \rceil - 2m_2 + 2i + 2$ and $4m' + m_1' + \ldots + m_n' = 4m - 4m_2 + 4i + 4 + a_{m_2-i}$.

$$
\begin{aligned}
s_d =& v_d - \frac{1}{2}((d'+1)(d'+2) - 4m'(m'+1) - \sum_{i=0}^{n} m_i'(m_i'+1)) \\
&+ \frac{3}{2}((d'-2)(d'-1) - 4(m'-1)m' - \sum_{i=1}^{n}(m_i'-1)m_i') \\
&- \frac{3}{2}((d'-3)(d'-2) - 4(m'-1)m' - \sum_{i=1}^{n}(m_i'-1)m_i') \\
&+ \frac{1}{2}((d'-4)(d'-3) - 4(m'-1)m' - \sum_{i=1}^{n}(m_i'-1)m_i') \\
=& v_d + \frac{1}{2}(-3d' - 9d' + 15d' - 7d' - 2 + 6 - 18 + 12 + 4m' \\
&+ \sum_{i=1}^{n} m_i' + 3(4m' + \sum_{i=1}^{n} m_i') - 3(4m' + \sum_{i=1}^{n} m_i') + 4m' + \sum_{i=1}^{n} m_i') \\
=& v_d - 2d' - 1 + 4m' + \sum_{i=1}^{n} m_i' \\
=& v_d - 4m - 2\lceil \frac{a_{m_2-i}}{2} \rceil + 4m_2 - 4i - 4 - 1 + 4m - 4m_2 + 4i + 4 + a_{m_2-i} \\
=& v_d - 2\lceil \frac{a_{m_2-i}}{2} \rceil - 1 + a_{m_2-i} = 0,
\end{aligned}
$$

in any of the two cases, because if a_{m_2-i} is even, then $2 \cdot \lceil \frac{a_{m_2-i}}{2} \rceil = a_{m_2-i}$ and $v_d = 1$, and if a_{m_2-i} is odd, then $2 \cdot \lceil \frac{a_{m_2-i}}{2} \rceil = a_{m_2-i} + 1$ and $v_d = 2$.

Combining all the results, we finally get Theorem B.

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