Dirac geometry and moment maps

Eckhard Meinrenken
(notes taken by Sven Porst and Christoph Wockel)

June 5, 2009

I Dirac geometry and moment maps

I.1 Linear Dirac Structures

V: vectors space, $\mathbb{V} := V \otimes V^*$ carries $\langle (v_1, \alpha_1), (v_2, \alpha_2) \rangle = \alpha_1(v_2) + \alpha_2(V_1)$.

Definition I.1. A morphism $(f, \omega) : \mathbb{V} \to \mathbb{V}$ is a linear map $f : V \to V'$ and $\omega \in \Lambda^2 V^*$. Composition is defined by

$$(f', \omega') \circ (f, \omega) = (f' \circ f, \omega + f^* \omega')$$ ■

Definition I.2. $(v, \alpha) \sim_{(f,\omega)} (v', \alpha') \Leftrightarrow v' = f(v)$ and $\alpha = f^* \alpha' + \iota_v \omega$. ■

Then: $x_1 \sim x_1'$, $x_2 \sim x_2' \Rightarrow \langle x_1, x_2 \rangle = \langle x_1', x_2' \rangle$

$$\ker(f, \omega) := \{x \in \mathbb{V} : x \sim_{f,\omega} 0\}$$

Definition I.3. A Dirac structure on V is a maximal isotropic subspace $E \subseteq \mathbb{V}$. A Dirac morphism $(f, \omega) : (\mathbb{V}, E) \to (\mathbb{V}', E')$ is a morphism (f, ω) s.th.

1. $E' = \{x' \in \mathbb{V}' : \exists x \in E, x \sim_{(f,\omega)} x'\}$
2. $E \cap \ker(f, \omega) = 0$

I.e., every $x' \in E'$ comes from a unique $x \in E$. ■

Example I.4.
- Any morphism (f, ω) defines a Dirac morphism.
- The morphism $(f, \omega) : (\mathbb{V}, V) \to (0, 0)$ is a Dirac morphism $\Leftrightarrow$ ω is symplectic. ■

$\rightsquigarrow$ clear how to define Dirac structures on vector bundles $V \to M$

Example I.5. X: Euclidean vector space ($X \cong X^*$). Then we get a map

$$O(X) \to \mathrm{Lag}(\mathbb{X}), \quad A \mapsto E_A := \{e_A(\xi), \xi \in X\}$$

with $e_A(\xi) = ((I - A^{-1})\xi, (\frac{I + A^{-1}}{2})\xi)$. ■

Let $V = O(X) \times X \to O(X)$ be the trivial vector bundle. This carries a "tautological" Dirac structure; $E|_A = E|_A \subseteq X \otimes X^*$.

$$\exists \text{Dirac morphism} (f, \sigma) : (V, E) \times (V, E) \to (V, E)$$

covering the multiplication map $O(X) \times O(X) \to O(X)$.

I.2 Dirac stuctures on Manifolds

Definition I.6. A Dirac structure (M, E, η) on M is a smooth linear Dirac structure E on TM, together with a closed 3-form η s.th. $\Gamma(E)$ is closed under

$$[(X,\alpha),(Y,\beta)]_E = ([X,Y], L_X\beta - \iota_Y d\alpha - \iota_Y\iota_X\eta)$$

(the Courant Bracket or Dorfmann Bracket). This is in fact a Leibnitz bracket! ■

Then $(E, [\cdot\,,\cdot]_E, \mathrm{per}_{TM})$ is a Lie algebroid $\rightsquigarrow$ $\mathrm{pr}_{TM} E \subseteq TM$ gives a foliation.

Definition I.7. A Dirac morphism $(\Phi,\omega) : (M, E, \eta) \to (M', E', \eta')$ is given by a smooth map $\Phi : M \to M$ ans $\omega \in \Omega^2(M)$ s.th. $(d\Phi, \omega)$ is a linear Dirac morphism $(\mathbb{T}M, E) \to (\mathbb{T}M', E')$ and $\eta = \Phi^*\eta' + d\omega$. ■

Example I.8.

- (M, T^*M, η) defines a Dirac structure
- $(M, TM, 0)$ defines a Dirac structure

A more interesing example is: $\mathfrak{g}$ Lie algebra. Then $(\mathfrak{g}^*, E_{\mathfrak{g}^*}, 0)$ defines a Dirac structure, where

$$E_{\mathfrak{g}^*}\big|_\mu = \{(\xi_{\mathfrak{g}^*}, \langle d\mu, \xi\rangle) | \xi \in \mathfrak{g}\} \subseteq T_\mu\mathfrak{g}^* \otimes T^*_\mu\mathfrak{g}^*$$

$$\begin{aligned}&\mathrm{Add} : \mathfrak{g}^* \to \mathfrak{g}^* (\text{addition})\\ &(\mathrm{Add}, 0) : (\mathfrak{g}^*, E_{\mathfrak{g}^*}, 0) \times (\text{...same...}) \to (\text{...same...})\end{aligned}$$ ■

Example I.9. G : Lie group with invariant inner product on $\mathfrak{g} = \mathrm{Lie}(G)$. Then

$$\eta = \frac{1}{12}\theta^l \cdot [\theta^l, \theta^l] \in \Omega^3(G), \text{ where } \omega^l = dg^{-1} \in \Omega^1(G, \mathfrak{g})$$

Then (E, E_G, η) is a Dirac structure, where

$$E_G\big|_{\mathfrak{g}} = \{(\xi^\#(g), \frac{\theta^l + \theta^r}{2}\xi) | \xi \in \mathfrak{g}\}$$

with $\mu : G \times G \to G$ and $\sigma = \frac{1}{2}\,\mathrm{pr}_1^*(\theta^l) \cdot \mathrm{pr}_2^*\,\theta^r$ we obtain a Dirac morphism

$$(\mu, \sigma) : (G, E_G, \eta) \times (G, E_G, \eta) \to (G, E_G, \eta)$$ ■

I.3 Moment maps

Fact:

$$(\Phi, \omega) : (M, TM, 0) \to (\mathfrak{g}, E_{\mathfrak{g}^*}, 0) \Leftrightarrow M \text{ is a Hamiltonian } \mathfrak{g}\text{-space}$$

that means that

- $\mathfrak{g}$ acts on M, $\omega \in \Omega^2(M)$, $\Phi : M \to \mathfrak{g}^*$ are equivariant
- $d\omega = 0$
- $\iota(\xi_M)\omega = -\langle d\Phi, \xi\rangle$ for $\xi \in \mathfrak{g}$
- $\ker(\omega) = 0$

Fact: $(\Phi, \omega) : (M, TM, 0) \to (G, E_G, \eta)$ is a Dirac morphism $\Leftrightarrow$ (M, ω, Φ) is a q-Hamiltonian $\mathfrak{g}$-space. That means

- $\mathfrak{g} \curvearrowright M$ such that ω, Φ are equivariant
- $d\omega = -\Phi^*\eta$
- $\iota(\xi_M)\omega = -\Phi^*(\frac{\theta^l+\theta^r}{2})\xi$
- $\ker(\omega) \cap \ker(d\Phi) = 0$

One calls this a q-Hamiltonian G-space if the action integrates to a G-action.

Table of analogies:

Hamiltonian	q-Hamiltonian
$\mathcal{O} \subseteq \mathfrak{g}^*$ coadjoint orbit	$\mathcal{C} \hookrightarrow G$ (conj. class)
$T^*G \cong G \times \mathfrak{g}^*$	$D(G) = G \times G$; $\Phi(a,b) = aba^{-1}b^{-1}$
$\mathrm{SU}(n) \curvearrowright \mathbb{C}^n = \mathbb{R}^{2n}$	$\mathrm{SU}(n) \curvearrowright S^{2n}$
$\mathrm{Sp}(n) \curvearrowright \mathbb{H}^n$	$\mathrm{Sp}(n) \curvearrowright \mathbb{H}P(n)$
products by comp. with $(\mathrm{Add}, 0)$	products by comp. with (μ, σ)
symplectic reduction $M//\Phi^{-1}(0)/G$	$M//G = \Phi^{-1}(e)/G$
convexity, thus DH-theory	$\cdots$ (similar, but more tricky)
intersection pairing on $M//G$ via localization on M	$\cdots$ (similar, but more tricky)
Kirwan surjectivity results	$\cdots$ (similar, but more tricky)
prequantization, quantization	$\cdots$ (similar, but more tricky)

II Dixmier-Douady Theory and twisted K-theory

$\mathcal{H}$: complex holbert space (finite- or infinte-dimensional), $\mathbb{L}(\mathcal{H})$: bounded linear operators, $\mathbb{K}(\mathcal{H}) \subseteq \mathbb{B}(\mathcal{H}) = \overline{\mathbb{L}_{\text{fin.}}(\mathcal{H})}$ C^*-algebra of compact operators.

Fact: $\mathrm{Aut}(\mathbb{K}(\mathcal{H})) = \mathrm{PU}(\mathcal{H})$ (w.r.t the strong operator topology)

Definition II.1. A *Dixmire–Douady bundle* (or (DD) bundle) over X is a bundle of $*$-algebras $\mathcal{A} \to X$ with typical fiber $\mathbb{K}(\mathcal{H})$ and structure group $\mathrm{PU}(\mathcal{H})$. ■

Varaiants: G-equivariant bundles or $\mathbb{Z}_2$-graded

Example II.2. If M is an even-dimensional Riem. manifold $\Rightarrow \mathbb{C}l(M)$ (the Clifford bundle) is a (DD) bundle. ■

Recall: $\mathbb{C}l(\mathbb{R}^{2n})$ has generators e_i and relations $e_ie_j + e_je_i = 2\delta_{ij}$. Moreover, it is a matrix algebra isomorphic to $\mathrm{End}(\mathbb{C}^n)$.

Definition II.3. A *Morita morphism* $\mathcal{E} : \mathcal{A}_1 \to \mathcal{A}_2$ is a Banach bundle $\mathcal{E} \to X$ (all over the same base X) with a bimodule structure

$$\mathcal{A}_2 \curvearrowright \mathcal{E} \curvearrowleft \mathcal{A}_1,$$

locally modeled on $\mathbb{K}(\mathcal{H}_2) \curvearrowright \mathbb{K}(\mathcal{H}_1, \mathcal{H}_2) \curvearrowleft \mathbb{K}(\mathcal{H}_1)$. ■

A *Morita trivialization* $\mathbb{C} \to \mathcal{A}$ is then given by a Hilber space bundle $\mathcal{E}$, together with an isomorphism $\mathcal{A} \cong \mathbb{K}(\mathcal{E})$.

Example II.4. ($\mathbb{Z}_2$-graded): A Morita trivialization of $\mathbb{C}l(M)$ is the same thing as a ($\mathbb{Z}_2$-graded) spinor bundle $S \to M$ (and then $\mathbb{C}l(M) \cong \mathrm{End}(S)$). This in turn is the same thing as a Spin_c-structure on M. ■

Definition II.5. An isomorphism of two Morita isomorphisms $\mathcal{E}, \mathcal{E}' : \mathcal{A}_1 \to \mathcal{A}_2$ is called a 2-isomorphism. ■

Fact: Any two such $\mathcal{E}, \mathcal{E}'$ are related by a line bundle

$$L = \mathrm{Hom}_{\mathcal{A}_1\text{-}\mathcal{A}_2}(\mathcal{E}, \mathcal{E}') \Leftrightarrow \mathcal{E}' \cong \mathcal{E} \otimes L$$

Theorem II.6 (Dixmier–Douady). *(DD) bundles up to Morita isomorphism over X are in bijection to $H^3(X, \mathbb{Z})$.*

Thus $DD(\mathcal{A}) \in H^3(X, \mathbb{Z})$ is an obstruction to the existence of Morita trivialization. In particular, $DD(\mathbb{C}L(M)) = w^3(M) \in H^3(M, \mathbb{Z})$ is the third integral Stiefel–Whitney class (???).

$$\begin{array}{ccc} \mathcal{A}_1 & \dashrightarrow & \mathcal{A}_2 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Phi} & X_2 \end{array}$$

(DD)-morphism

$$\begin{array}{rcc} (\Phi, \mathcal{E}) : \mathcal{A}_1 & \dashrightarrow & \mathcal{A}_2 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Phi} & X_2 \end{array}$$

is a Morita morphism $\mathcal{E} : \mathcal{A}_1 \to \Phi^* \mathcal{A}_2$

II.1 Twisted K-homology

$C(X)$: continuous functions on X, vanishing at $\infty \rightsquigarrow$ a C^*-algebra.

Definition II.7 (Atiyah, Kasparov). A Fredholm module over $C(X)$ is a triple $(\mathcal{H}, \pi, F)$, where $\mathcal{H}$ is $\mathbb{Z}_2$-graded Hilbert space, $\pi : C(X) \to \mathbb{L}(\mathcal{H})$ is an even $*$-morphism and $F \in \mathbb{L}(\mathcal{H})$ odd s.th. $\forall a \in C(X)$ we have

$$\begin{aligned} [\pi(a), F] &\in \mathbb{K}(\mathcal{H}) \\ (F - F^*)\pi(a) &\in \mathbb{K}(\mathcal{H}) \\ (F^2 - I)\pi(a) &\in \mathbb{K}(\mathcal{H}) \end{aligned}$$

■

Definition II.8.

ToDo is a covariant functor w.r.t proper maps. ■

Example II.9.

- $K_0^G(\mathrm{pt}) = \mathrm{Rep}(G)$ (with the ring structure given by $\mathrm{pt} \times \mathrm{pt} \to \mathrm{pt}$)
- $E = E^+ \oplus E^- \to M$ a $\mathbb{Z}_2$-graded hermitian vector bundle, $D \curvearrowright \Gamma(E)$ an odd, self-adjoint elliptic differential operator $\Rightarrow (\Gamma_{L^2}(M, E), \pi, \frac{D}{(1+D^2)''^2})$ is a Fredholm module $\rightsquigarrow$ $[D] \in K_0^G$. For $p : M \to \mathrm{pt}$, $\mathrm{ind_G(D)} = p_*[D] \in K_0^G(\mathrm{pt}) = \mathrm{Rep}(G)$.

II.2 Twisted K-homology

For a ($\mathbb{Z}_2$-graded) (DD) bundle $\mathcal{A} \to X$ define $K_0(X, \mathcal{A})$ as before, just replace $C(X)$ by $\Gamma(X, \mathcal{A})$.

Example II.10. $\mathbb{C}l(M)$ has two action on $\mathbb{C}l(TM) \cong \Lambda T^*M$ (by left- and right multiplication). The Dirac operator corresponding to the left action is the "deRham–Dirac" operator $D = d + d^*$. Take $\pi : \Gamma(M, \mathbb{C}l(TM)) \curvearrowright \Gamma_{L^2}(E)$ using the right action.

$$(\Gamma_{L^2}(E), \pi, \frac{D}{(1 + D^2}){''^2}) \rightsquigarrow [M] \in K_0(M, \mathbb{C}l(TM))$$

(the Kasparov fundamental class, playing the role of the fundamental class of a non-oriented manifold (with values in the orientation bundle)) ■

Example II.11. G: compact simple, simply connected Lie group $\Rightarrow H^i(G,\mathbb{Z}) = \mathbb{Z}$ if $i = 3$ and is trivial if $i = 1, 2$. Let $\mathcal{A} \to G$ be a DD bundle corresponding to $1 \in \mathbb{Z}$. Then $K_0^G(G, \mathcal{A}^l)$ carries a ring structure, defined by multiplication $G \times G \to G$ ■

Theorem II.12 (Freed-Hopkins-Teleman).

$$K_0^G(G, \mathcal{A}^{k+h^\vee}) = R_k(G) \text{ (the level } k \text{ fusion ring)} \tag{1}$$

$\forall k = 0, 1, 2, \ldots$, where $H^\vee$ is the dual Coxeter number of G.

What is $R_k(G)$? Choose $T \subseteq G$ a maximal torus, $\Lambda \subseteq \mathfrak{t} \subseteq \mathfrak{g}$ integral lattice, $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z}) \subseteq \mathfrak{t}^* \cong \mathfrak{t}$. Identify $\mathfrak{t}^* \cong \mathfrak{t}$ using the basic inner product.

$$T_l := \frac{1}{l}\Lambda^*/\Lambda \subseteq \mathfrak{t}/\Lambda = T$$

Then one may view $R(G)$ as characters and consider the vanishing ideal $I_k(G)$ for $T_{k+h^\vee} \cap G^{\mathrm{reg}}$ $\rightsquigarrow R_k(G) = R(G)/I_k(G)$. ■

Example II.13. $G = \mathrm{SU}_2$, $R(G) = \mathbb{Z}[\chi_0, \chi_1, \chi_2, \ldots]\ldots$ $I_k(G) =< \chi_{k+1} >$, $R(G)/I_k(G) = \mathbb{Z}[\tau_0, \ldots, \tau_k]$ with $\tau_i =$ images of χ_i. ■

III Quantization of group-valued moment maps

I missed this lecture for I had to teach. Here are the scanned notes of Sven Porst.

HAMILTONIAN G-SPACES

$(M, \Phi : M \to \mathfrak{g}^*, \omega \in \Omega^2 M)$

$d\omega = 0$
$\ker \omega = 0$
+ moment map condition

Step 1 Compatible almost complex structure
$\longrightarrow$ Spin$_c$ structure

$S : \mathbb{C}l(TM) \dashrightarrow \mathbb{C}$

Step 2 (Prequantisation)
Choose prequantum line bundle $\begin{smallmatrix} L \\ \downarrow \\ M \end{smallmatrix}$
$\mathrm{curv}(L) = \omega$

Consider Spin$_c$-Dirac operator $\partial\!\!\!/_L$ for Spin$_c$-structure $S \otimes L$

Define quantisation
$Q(M) = \mathrm{index}_G(\partial\!\!\!/_L) \in R(G)$

Properties:
$Q(M_1 \cup M_2) = Q(M_1) + Q(M_2)$
$Q(M_1 \times M_2) = Q(M_1)\, Q(M_2)$
$Q(M^*) = Q(M)^*$ $\quad M^* = (M, -\omega, -\Phi)$
$Q(M/\!/G) = Q(M)^G$:= multiplicity of trivial rep
$Q(\mathcal{O}) = \chi_{\mathcal{O}}$ character of irrep given by $\mathcal{O} \subset \mathfrak{g}^*$ coadj. orbit (coadjoint orbit)

Q-HAMILTONIAN G-SPACES

$(M, \Phi : M \to G, \omega \in \Omega^2 M)$

$d\omega = -\Phi^* \eta$
$\ker \omega \cap \ker d\Phi = 0$
+ moment map cond
[G cpct $\pi_1 G = 0$]

Step 1: Have distinguished Morita morphism
$(\Phi, S) : \mathbb{C}l(TM) \dashrightarrow \mathcal{A}_G^{\mathrm{spin}}$

Step 2: Prequantisation amounts to integral lift of $[\omega] \in H^2_{dR}(M) \leftarrow H^2(M, \mathbb{Z})$

$d\omega = -\Phi^* \eta$, $d\eta = 0$ mean that (ω, η) define relative cocycle in $\Omega^3_{dR}(\Phi)$.
Recall: $\Phi\, M \to N$ $\quad H^\bullet(\Phi) := H^\bullet(\mathrm{cone}_\Phi)$
can be computed using
$C^\bullet(\Phi) = C^{\bullet-1}(M) \oplus C^\bullet(N)$
$d(\alpha, \beta) = (d\alpha + \Phi^* \beta, -d\beta)$

(Prequantisation)
Choose integral lift of
$[(\omega, \eta)] \in H^3_{dR}(\Phi)$

Def. A level k prequantisation of (M, ω, Φ) is an integral lift of $k[(\omega, \eta)]$.

<u>Thm</u> A level k prequantisation gives Morita morphism.

$$(\Phi, \mathcal{E}): M \times \mathbb{C} \dashrightarrow \mathcal{A}^k_G$$

So for Step 2 in q-Hamiltonian case have

level k prequantisation $\leadsto (\Phi, \mathcal{E}): M \times \mathbb{C} \dashrightarrow \mathcal{A}^k_G$

Tensor these together to get

$$(\Phi, S \otimes \mathcal{E}): \mathbb{C}l(TM) \dashrightarrow \mathcal{A}^{k+h^\vee}_G.$$

This defines push forward

$$K^G_0(M, \mathbb{C}l(TM)) \longrightarrow K^G_0(G, \mathcal{A}^{k+h^\vee}_G) = R_k(G)$$

$[M] \in K^G_0(M, \mathbb{C}l(TM))$; $R_k(G)$ ← level k Verlinde algebra

<u>Def</u> $Q(M) := \Phi_*([M]) \in R_k(G)$

The properties of this are similar to those of the quantisation of Hamiltonian G-spaces.

$$Q(M_1 \times M_2) = Q(M_1)\, Q(M_2) \quad \text{etc}$$

$$Q(M/\!/G) = Q(M)^G \quad \text{multiplicity of } \tau_0 \in R_k(G) = R(G)/I_k(G).$$

(Application: Moduli spaces of flat connections.)

<u>Examples:</u>

$D(G) = G \times G \quad \Phi(a,b) = aba^{-1}b^{-1}$ is prequantisable $\forall k \in \mathbb{N}$

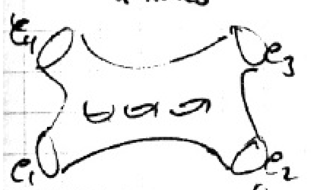
$SU(n) \circlearrowright S^{2k}$; $Sp(n) \circlearrowright \mathbb{H}P(n)$ is prequantisable $\forall k \in \mathbb{N}$

$\mathcal{C} = G\cdot\exp(\xi)$ $\quad \xi \in \mathfrak{t} \subset \mathfrak{g}$ $\quad$ is prequantisable at level $k \in \mathbb{N} \iff \xi \in \frac{1}{k}\Lambda^*$

$\Lambda \subset \mathfrak{t}$ integral lattice
$\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z}) \subset \mathfrak{t}^* = \mathfrak{t}$

$$\underbrace{D(G)\times \dots \times D(G)}_{h \text{ times}} \times \mathcal{C}_1 \times \dots \times \mathcal{C}_r /\!\!/ G = M(\Sigma_h^r;\ \mathcal{C}_1, \dots, \mathcal{C}_r)$$

(r: boundary components; h: genus)



$\rightsquigarrow$ Get a formula for the quantisation of this moduli space „Verlinde numbers"