Professor Dirac's Cookbook

or

How to Construct a Dirac Operator in Infinite Dimensions

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In Today's Program

Recipe of the Day

Cooking up a Dirac operator

Ingredients Under the Microscope

Orthogonal structures

Grow Your Own

co-Riemannian structure

Recipe of the Day

The Basic Recipe

Ingredients

Remarks

Method

The Basic Recipe

Ingredients

1. Smooth manifold, M

Remarks

oriented, spin, even dimensional, ...

Method

The Basic Recipe

Ingredients

1. Smooth manifold, M
2. Riemannian structure, g

Remarks

$$g_p : T_p M \times T_p M \to \mathbb{R}$$

$$g_p : T_p M \xrightarrow{\cong} T_p^\star M$$

Method

The Basic Recipe

Ingredients

1. Smooth manifold, M
2. Riemannian structure, g

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Method

1. Add g to M and leave until a connection ω appears.

The Basic Recipe

Ingredients

1. Smooth manifold, M
2. Riemannian structure, g
3. Spin structure, Q

Remarks

$$\begin{array}{ccccc} \mathbb{Z}_2 & \to & \mathrm{Spin}_n & \to & \mathrm{SO}_n \\ \mathbb{Z}_2 & \to & Q & \to & P \end{array}$$

Method

1. Add g to M and leave until a connection ω appears.

The Basic Recipe

Ingredients

1. Smooth manifold, M
2. Riemannian structure, g
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Remarks

Method

1. Add g to M and leave until a connection ω appears.
2. Place Q over the mixture and allow ω to infuse upwards.

The Basic Recipe

Ingredients

1. Smooth manifold, M
2. Riemannian structure, g
3. Spin structure, Q
4. Spin representations, $S^{\pm}$

Remarks

Has Clifford Multiplication
$c : \mathbb{R}^n \times S^+ \to S^-$ (bilinear)

Method

1. Add g to M and leave until a connection ω appears.
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The Basic Recipe

Ingredients

1. Smooth manifold, M
2. Riemannian structure, g
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4. Spin representations, $S^{\pm}$

Remarks

Clifford Multiplication becomes

$$c : TM \times S^+_M \to S^-_M$$

(fibrewise bilinear)

Method

1. Add g to M and leave until a connection ω appears.
2. Place Q over the mixture and allow ω to infuse upwards.
3. Combine $S^{\pm}$ with Q to produce bundles $S^{\pm}_M$.

The Basic Recipe

Ingredients

1. Smooth manifold, M
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Method

1. Add g to M and leave until a connection ω appears.
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The Basic Recipe

Ingredients

1. Smooth manifold, M
2. Riemannian structure, g
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Remarks

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\begin{aligned}
\partial\!\!\!/ : \Gamma(S_M^+) &\xrightarrow{\nabla} \Gamma(\mathscr{L}(TM, S_M^+)) \\
&\xrightarrow{\cong} \Gamma(T^\star M \otimes S_M^+) \\
&\xrightarrow{g^{-1}} \Gamma(TM \otimes S_M^+) \\
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\end{aligned}
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Method

1. Add g to M and leave until a connection ω appears.
2. Place Q over the mixture and allow ω to infuse upwards.
3. Combine $S^{\pm}$ with Q to produce bundles $S_M^{\pm}$.
4. Apply ω to S_M^+ to produce a covariant differential operator ∇.
5. Combine ∇, g, and c to produce the Dirac operator $\partial\!\!\!/$.

First Variation

Ingredients

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Remarks

infinite dimensional,
polarised, oriented, spin ...

Method

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$$g_p : T_pX \times T_pX \to \mathbb{R}$$
$$g_p : T_pX \to T_p^\star X$$

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Remarks

$$\begin{array}{ccccc} S^1 & \to & \mathrm{Spin}_J & \to & \mathrm{SO}_J \\ S^1 & \to & Q & \to & P \end{array}$$

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First Variation

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Remarks

No longer canonical

Method

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Remarks

Has Clifford Multiplication
$c : V \times S^{+} \to S^{-}$ (bilinear)

Method

1. Add g to X and leave until a connection ω appears.
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First Variation

Ingredients

1. Smooth manifold, X
2. Riemannian structure, g
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4. Spin representations, $S^{\pm}$

Remarks

Clifford Multiplication becomes

$$c : TX \times S_X^+ \to S_X^-$$

(fibrewise bilinear)

Method

1. Add g to X and leave until a connection ω appears.
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First Variation

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Remarks

$$\not{\partial} : \Gamma(S_X^+) \xrightarrow{\nabla} \Gamma(\mathscr{L}(TX, S_X^+))$$
$$\to \Gamma(T^\star X \otimes S_X^+)$$
$$\xrightarrow{g^{-1}} \Gamma(TX \otimes S_X^+)$$
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Method

1. Add g to X and leave until a connection ω appears.
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Result

Total collapse of Dirac soufflé.

The Collapse

$$\Gamma(S_X^+) \xrightarrow{\nabla} \Gamma(\mathscr{L}(TX, S_X^+))$$

$$\uparrow \quad f \otimes v \mapsto (u \mapsto f(u)v)$$

$$\Gamma(T^\star X \otimes S_X^+)$$

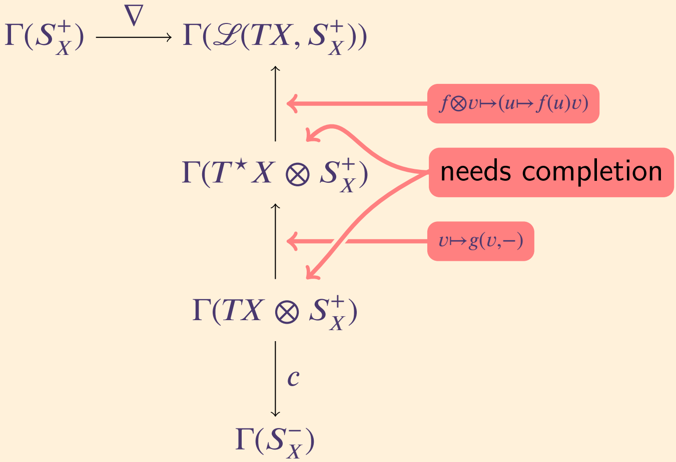
$$\uparrow \quad v \mapsto g(v,-)$$

$$\Gamma(TX \otimes S_X^+)$$

$$\downarrow c$$

$$\Gamma(S_X^-)$$

The Collapse



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$$\Gamma(TX \,\widehat{\otimes}\, S_X^+)$$

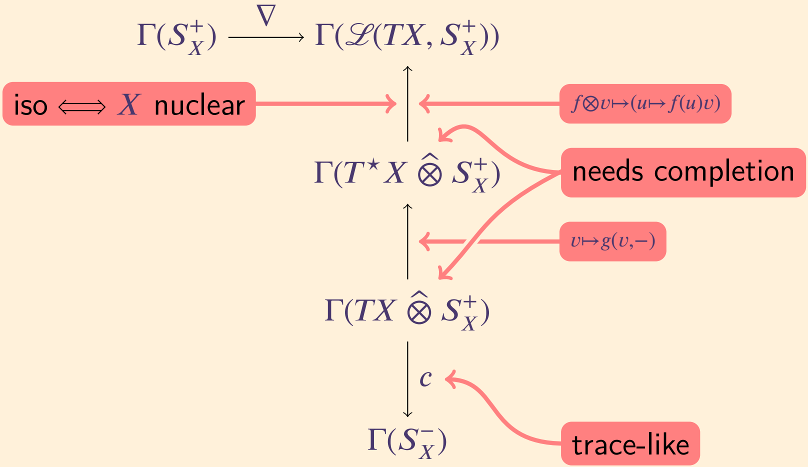
$$\downarrow c$$

$$\Gamma(S_X^-)$$

needs completion

trace-like

The Collapse



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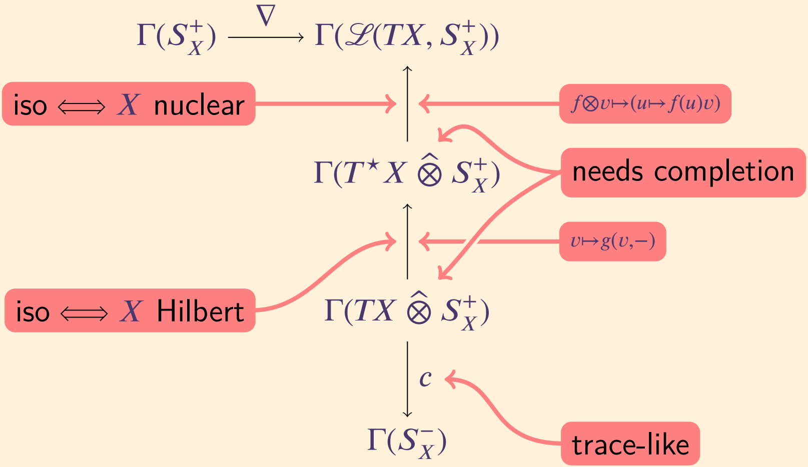

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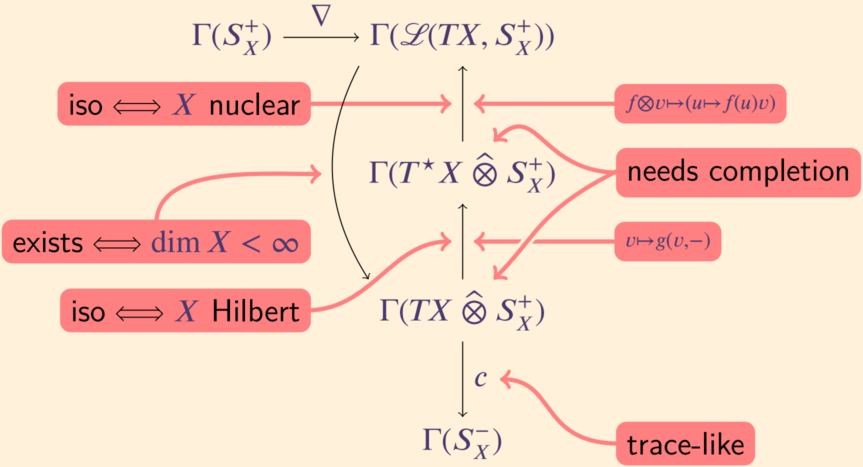

The Improved Basic Recipe

Ingredients

Remarks

Method

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Ingredients

1. Smooth manifold, M

Remarks

oriented, spin, even dimensional, ...

Method

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$$g_p : T_p M \times T_p M \to \mathbb{R}$$

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1. Add g to M and leave until a connection ω appears.

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Remarks

$$\begin{array}{ccccc} \mathbb{Z}_2 & \to & \mathrm{Spin}_n & \to & \mathrm{SO}_n \\ \mathbb{Z}_2 & \to & Q & \to & P \end{array}$$

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The Improved Basic Recipe

Ingredients

1. Smooth manifold, M
2. Riemannian structure, g
3. Spin structure, Q
4. Spin representations, $S^{\pm}$

Remarks

Build from $\mathbb{R}^{n\star}$

Has Clifford Multiplication

$c : \mathbb{R}^{n\star} \times S^{+} \to S^{-}$ (bilinear)

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(fibrewise bilinear)

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Remarks

$$\begin{aligned} \partial\!\!\!/ : \Gamma(S_M^+) &\xrightarrow{\nabla} \Gamma(\mathscr{L}(TM, S_M^+)) \\ &\xrightarrow{\cong} \Gamma(T^\star M \otimes S_M^+) \\ &\xrightarrow{c} \Gamma(S_M^-) \end{aligned}$$

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The Improved Basic Recipe

Ingredients

1. Smooth manifold, M
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Remarks

Isomorphism

$g : TM \cong T^{\star}M$

gives equivalence

Method

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Second Variation

Ingredients

Remarks

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Remarks

infinite dimensional,
polarised, oriented, spin ...

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$$g_p : T_pX \times T_pX \to \mathbb{R}$$
$$g_p : T_pX \to T_p^\star X$$

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$$\begin{array}{ccccc} S^1 & \to & \mathrm{Spin}_J & \to & \mathrm{SO}_J \\ S^1 & \to & Q & \to & P \end{array}$$

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Second Variation

Ingredients

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Remarks

No longer canonical

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Second Variation

Ingredients

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Remarks

Build from $V^{\star}$

Has Clifford Multiplication

$c : V^{\star} \times S^{+} \to S^{-}$ (bilinear)

Method

1. Add g to X and leave until a connection ω appears.
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Second Variation

Ingredients

1. Smooth manifold, X
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Remarks

Clifford Multiplication
becomes
$c : T^{\star}X \times S_X^{+} \to S_X^{-}$
(fibrewise bilinear)

Method

1. Add g to X and leave until a connection ω appears.
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Second Variation

Ingredients

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Remarks

$$\partial\!\!\!/ : \Gamma(S_X^+) \xrightarrow{\nabla} \Gamma(\mathscr{L}(TX, S_X^+)) \longrightarrow \Gamma(T^\star X \otimes S_X^+) \xrightarrow{c} \Gamma(S_X^-)$$

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Question

Does it collapse?

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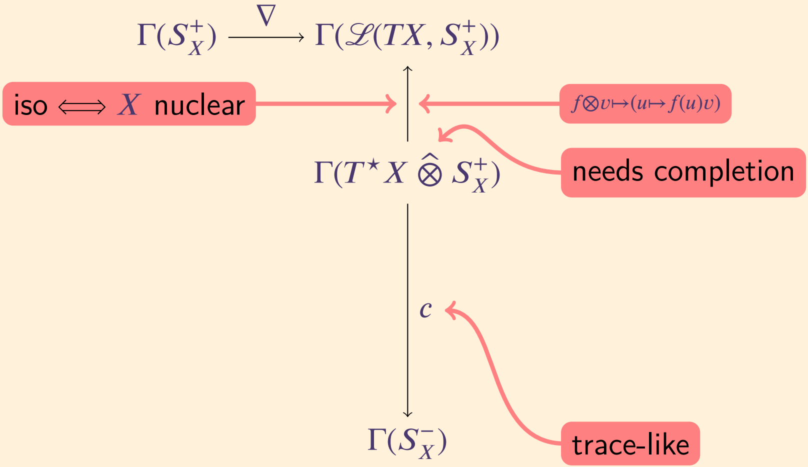
Question

Does it collapse?

Collapse?



$\Gamma(S_X^+) \xrightarrow{\nabla} \Gamma(\mathscr{L}(TX, S_X^+))$
iso $\iff$ X nuclear
$f \otimes v \mapsto (u \mapsto f(u)v)$
needs completion
$\Gamma(T^\star X \mathbin{\widehat{\otimes}} S_X^+)$
c
trace-like
$\Gamma(S_X^-)$

Collapse? Not if we're nuclear

$$\Gamma(S_X^+) \xrightarrow{\nabla} \Gamma(\mathscr{L}(TX, S_X^+))$$

$$\Gamma(T^\star X \mathbin{\widehat{\otimes}} S_X^+) \longrightarrow \Gamma(\mathscr{L}(TX, S_X^+))$$

$$\Gamma(T^\star X \mathbin{\widehat{\otimes}} S_X^+) \xrightarrow{c} \Gamma(S_X^-)$$

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Ingredients Under the Microscope

A Close Examination of the Ingredients

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Under the Microscope

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Under the Microscope

Remarks

Build from $V^\star$

Has Clifford Multiplication

$$c : V^\star \times S^+ \to S^-$$

$$c : T^\star X \times S^+_X \to S^-_X$$

A Close Examination of the Ingredients

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$$c : T^{\star}X \times S^{+}_{X} \to S^{-}_{X}$$

Under the Microscope

Construction of $S^{\pm}$ starts from

$$\mathrm{Cl}(W) := T(w) \big/ w \otimes w - g(w,w)1$$

A Close Examination of the Ingredients

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Under the Microscope

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Build from $V^{\star}$

Has Clifford Multiplication

$$c : V^{\star} \times S^{+} \to S^{-}$$

$$c : T^{\star}X \times S^{+}_{X} \to S^{-}_{X}$$

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Under the Microscope

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So need g on $V^{\star}$ not V

Remarks

Build from $(V^{\star}, g)$

Has Clifford Multiplication

$$c : V^{\star} \times S^{+} \to S^{-}$$

$$c : T^{\star}X \times S_X^{+} \to S_X^{-}$$

A Close Examination of the Ingredients

Ingredients

1. Smooth manifold, X
2. co-Riemannian structure, g
3. Spin structure, Q
4. Spin representations, $S^{\pm}$

Under the Microscope

Construction of $S^{\pm}$ starts from

$$\mathrm{Cl}(W) := T(w) \Big/ w \otimes w - g(w, w)1$$

So need g on $V^{\star}$ not V

Remarks

Build from $(V^{\star}, g)$

Has Clifford Multiplication

$$c : V^{\star} \times S^{+} \to S^{-}$$

$$c : T^{\star}X \times S^{+}_{X} \to S^{-}_{X}$$

A Close Examination of the Ingredients

Ingredients

1. Smooth manifold, X
2. co-Riemannian structure, g
3. Spin structure, Q
4. Spin representations, $S^{\pm}$

Under the Microscope

Construction of $S^{\pm}$ starts from

$$\mathrm{Cl}(W) := T(w) \Big/ w \otimes w - g(w, w)1$$

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Ratatouille

Gusteau:

"*Anyone can cook*"

or

Anton Ego:

"*No, I don't think anyone can*"

Weak or Strong?

Then the guard looks in politely and will ask you very brightly
"Do you like your morning tea weak or strong?"
But Skimble's just behind him and was ready to remind him,
For Skimble won't let anything go wrong. *T. S. Eliot*

Weak or Strong?

Then the guard looks in politely and will ask you very brightly
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Tea Through the Ages

1911: Weak or Strong

Weak or Strong?

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Tea Through the Ages

1911: Weak or Strong

2011: Black, green, chamomile, strawberry, jasmine, Earl Grey, chai, ... (but not Lipton)

Classifying Orthogonal Structures

Weak	$g_p : E_p \times E_p \to \mathbb{R}$	inner product space
Strong	$g_p : E_p \xrightarrow{\cong} E_p^\star$	Hilbert space

Classifying Orthogonal Structures

Weak	$g_p \colon E_p \to \overline{E}_p \to E_p^\star$	fibrewise Hilbert completion
Strong	$g_p \colon E_p \xrightarrow{\cong} E_p^\star$	Hilbert space

Classifying Orthogonal Structures

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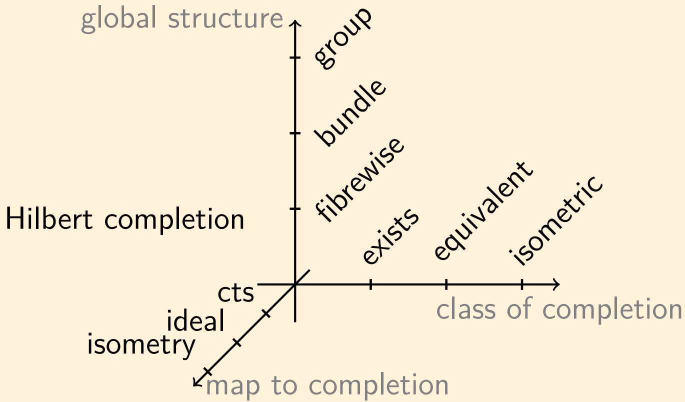


Grow Your Own

Loop Spaces: Common-or-Garden or Organic?

Question

1. Do loop spaces have orthogonal structures on their cotangent bundles?
2. If so, how good?

Loop Spaces: Common-or-Garden or Organic?

Question

1. Do loop spaces have orthogonal structures on their cotangent bundles?
2. If so, how good?

Answer

1. Yes
2. Good, but not quite as good as on the tangent bundle.

Grow Your Own Orthogonal Structure

$$LU_n$$

$$L\mathbb{C}^n \subseteq L^2\mathbb{C}^n \subseteq (L\mathbb{C}^n)^\star$$

In algebraic topological soil

Grow Your Own Orthogonal Structure

$$L_{\mathsf{pol}}U_n$$

$$\downarrow$$

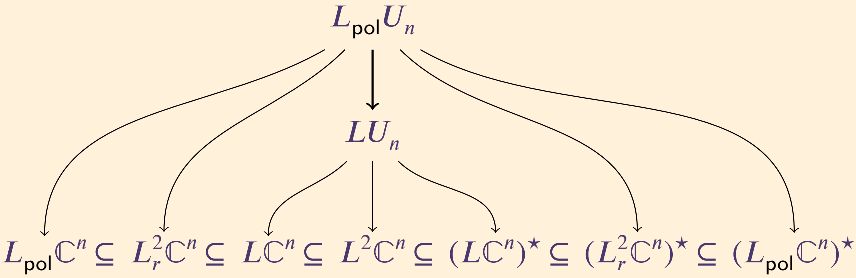
$$LU_n$$

$$\swarrow \quad \downarrow \quad \searrow$$

$$L\mathbb{C}^n \subseteq L^2\mathbb{C}^n \subseteq (L\mathbb{C}^n)^\star$$

In algebraic topological soil

Grow Your Own Orthogonal Structure



In algebraic topological soil

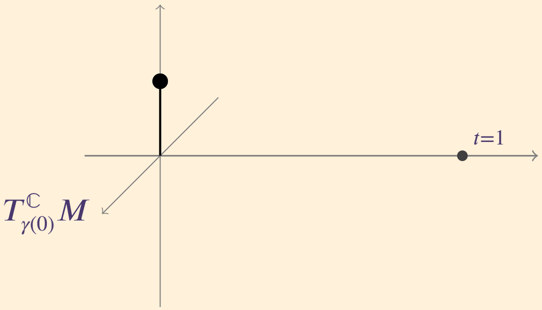
Grow Your Own Orthogonal Structure

$$L_{\mathsf{pol}}U_n \xrightarrow{\ \simeq\ } LU_n$$

$$L_{\mathsf{pol}}\mathbb{C}^n \subseteq L^2_r\mathbb{C}^n \subseteq L\mathbb{C}^n \subseteq L^2\mathbb{C}^n \subseteq (L\mathbb{C}^n)^\star \subseteq (L^2_r\mathbb{C}^n)^\star \subseteq (L_{\mathsf{pol}}\mathbb{C}^n)^\star$$

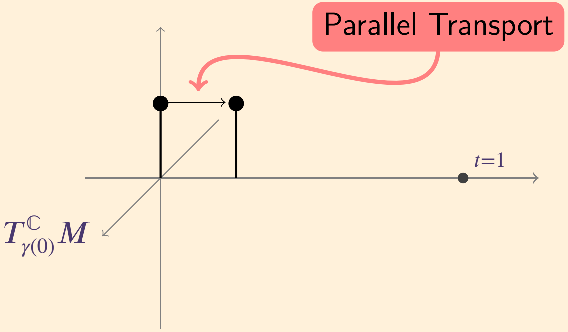
In algebraic topological soil

Grow Your Own Orthogonal Structure



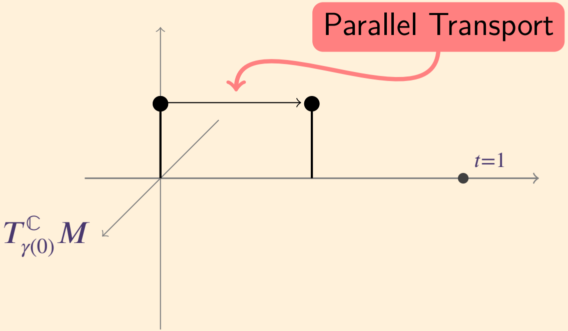
In differential soil

Grow Your Own Orthogonal Structure



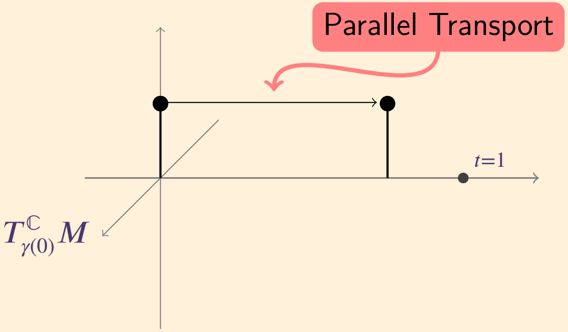
In differential soil

Grow Your Own Orthogonal Structure



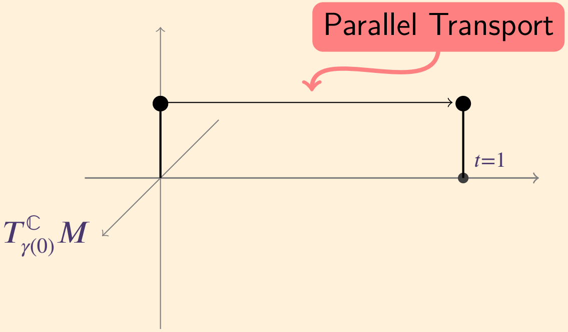
In differential soil

Grow Your Own Orthogonal Structure



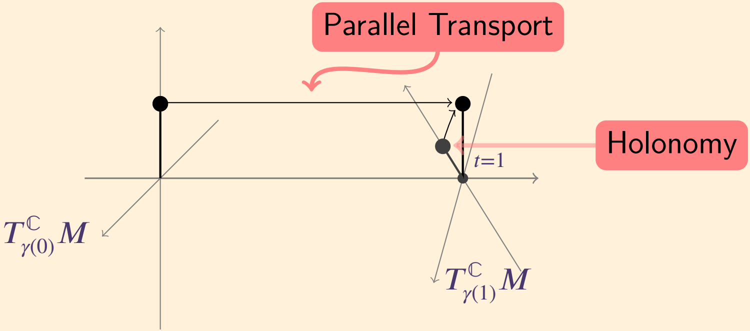
In differential soil

Grow Your Own Orthogonal Structure



In differential soil

Grow Your Own Orthogonal Structure



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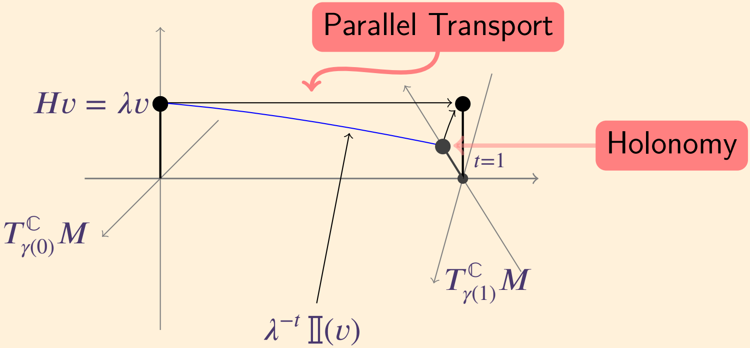
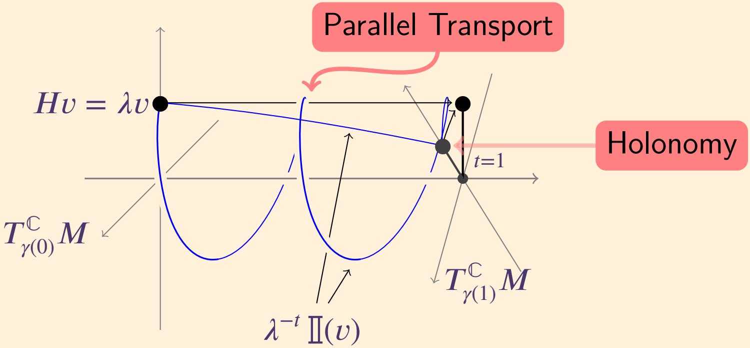


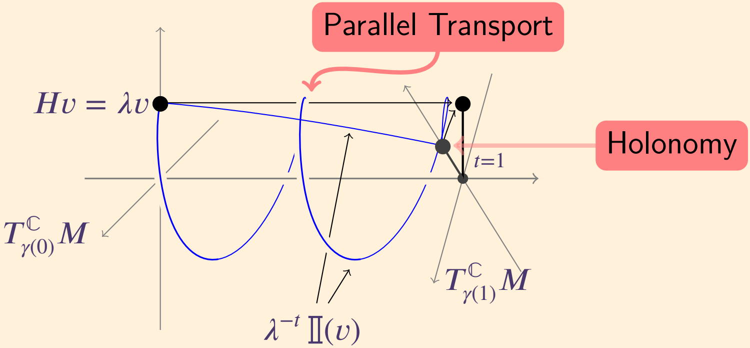
In differential soil

Grow Your Own Orthogonal Structure



In differential soil

Grow Your Own Orthogonal Structure



Any two choices differ by a polynomial

In differential soil

The Best Recipe

Ingredients

Remarks

Method

The Best Recipe

Ingredients

1. Smooth manifold, M
2. Riemannian structure, g

Remarks

oriented, spin, string, even dimensional, ...

Riemannian structure on M

Method

1. Add g to M and leave until a connection ω appears.

The Best Recipe

Ingredients

1. Smooth manifold, M
2. Riemannian structure, g

Remarks

$$g_p : T_pM \times T_pM \to \mathbb{R}$$
$$g_p : T_pM \xrightarrow{\cong} T_p^\star M$$

Method

1. Add g to M and leave until a connection ω appears.
2. Apply ω to TM to produce parallel transport $\mathbb{I}$.

The Best Recipe

Ingredients

1. Smooth manifold, M
2. Riemannian structure, g

Remarks

Method

1. Add g to M and leave until a connection ω appears.
2. Apply ω to TM to produce parallel transport $\mathbb{I}$.
3. Use $\mathbb{I}$ to extract $L_{\text{pol}}TM$ from LTM.

The Best Recipe

Ingredients

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Method

1. Add g to M and leave until a connection ω appears.
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3. Use $\mathbb{I}$ to extract $L_{\text{pol}}TM$ from LTM.
4. Add to $L^\star TM$ to get $(L_r^2 TM)^\star$.

The Best Recipe

Ingredients

1. Smooth manifold, M
2. Riemannian structure, g
3. Spin structure, Q

Remarks

$$\begin{array}{ccccc} S^1 & \to & \widetilde{L\,\mathrm{SO}_n} & \to & L\,\mathrm{SO}_n \\ S^1 & \to & Q & \to & P \end{array}$$

Method

1. Add g to M and leave until a connection ω appears.
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The Best Recipe

Ingredients

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Remarks

Method

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3. Use $\mathbb{I}$ to extract $L_{\text{pol}}TM$ from LTM.
4. Add to $L^\star TM$ to get $(L^2_r TM)^\star$.
5. Place Q over the mixture and allow $L\omega$ to infuse upwards.

The Best Recipe

Ingredients

1. Smooth manifold, M
2. Riemannian structure, g
3. Spin structure, Q
4. Spin representations, $S^{\pm}$

Remarks

Has Clifford Multiplication

$$c : L^{\star}\mathbb{R}^n \times S^+ \to S^-$$

(bilinear)

Method

1. Add g to M and leave until a connection ω appears.
2. Apply ω to TM to produce parallel transport $\mathbb{I}$.
3. Use $\mathbb{I}$ to extract $L_{\mathrm{pol}}TM$ from LTM.
4. Add to $L^{\star}TM$ to get $(L_r^2TM)^{\star}$.
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The Best Recipe

Ingredients

1. Smooth manifold, M
2. Riemannian structure, g
3. Spin structure, Q
4. Spin representations, $S^{\pm}$

Remarks

Clifford Multiplication becomes

$$c : T^{\star}LM \times S^{+}_{LM} \to S^{-}_{LM}$$

(fibrewise bilinear)

Method

1. Add g to M and leave until a connection ω appears.
2. Apply ω to TM to produce parallel transport $\mathbb{I}$.
3. Use $\mathbb{I}$ to extract $L_{\text{pol}}TM$ from LTM.
4. Add to $L^{\star}TM$ to get $(L^2_r TM)^{\star}$.
5. Place Q over the mixture and allow $L\omega$ to infuse upwards.
6. Combine $S^{\pm}$ with Q to produce bundles $S^{\pm}_{LM}$.

The Best Recipe

Ingredients

1. Smooth manifold, M
2. Riemannian structure, g
3. Spin structure, Q
4. Spin representations, $S^{\pm}$

Remarks

Method

1. Add g to M and leave until a connection ω appears.
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5. Place Q over the mixture and allow $L\omega$ to infuse upwards.
6. Combine $S^{\pm}$ with Q to produce bundles $S^{\pm}_{LM}$.
7. Apply $L\omega$ to S^{+}_{LM} to produce a covariant differential operator ∇.

The Best Recipe

Ingredients

1. Smooth manifold, M
2. Riemannian structure, g
3. Spin structure, Q
4. Spin representations, $S^{\pm}$

Remarks

$$\partial\!\!\!/ : \Gamma(S^+_{LM}) \xrightarrow{\nabla} \Gamma(\mathscr{L}(TLM, S^+_{LM}))$$

$$\xleftarrow{\cong} \Gamma(T^\star LM \hat{\otimes} S^+_{LM})$$

$$\xrightarrow{c} \Gamma(S^-_{LM})$$

Method

1. Add g to M and leave until a connection ω appears.
2. Apply ω to TM to produce parallel transport $\mathbb{I}$.
3. Use $\mathbb{I}$ to extract $L_{\text{pol}}TM$ from LTM.
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5. Place Q over the mixture and allow $L\omega$ to infuse upwards.
6. Combine $S^{\pm}$ with Q to produce bundles $S^{\pm}_{LM}$.
7. Apply $L\omega$ to S^+_{LM} to produce a covariant differential operator ∇.
8. Combine ∇ and c to produce the Dirac operator $\partial\!\!\!/$.

Result

A perfect Dirac soufflé.