THREE TROPICAL ENUMERATIVE PROBLEMS

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Abstract. In this survey, we describe three tropical enumerative problems and the corresponding moduli spaces of tropical curves. They have the structure of weighted polyhedral complexes. We observe similarities in the definitions of the weights, aiming at a better understanding of the tropical structure of the moduli spaces.

1. INTRODUCTION

In tropical geometry, algebraic varieties are degenerated to certain piece-wise linear objects called tropical varieties. This process loses a lot of information, but many properties of the algebraic variety can be read off the tropical variety, and many theorems that hold for the algebraic side remarkably continue to hold on the tropical side. Since tropical varieties are piece-wise linear, they are in principle easier to understand than algebraic varieties and combinatorial methods apply. Thus there is hope that we can use tropical geometry to derive theorems in algebraic geometry.

One of the fields in which tropical geometry has had significant success recently is enumerative geometry. Enumerative geometry deals with the counting of geometric objects that are determined by certain incidence conditions. The conditions have to be chosen in such a way that only finitely many objects satisfy them. We will consider tropical analogues of the following three examples of enumerative numbers:

1. The numbers $N(d,g)$ of nodal degree d genus g plane curves through $3d+g-1$ points in general position.
2. The numbers $E(d,j)$ of nodal degree d elliptic (that is, genus 1) plane curves and with fixed j-invariant j through $3d-1$ points in general position.
3. The Hurwitz numbers $H_d^g(\eta,\nu)$ of genus g, degree d covers of $\mathbb{P}^1$, with specified ramification profiles η and ν over 2 fixed points in $\mathbb{P}^1$ and at most simple ramification over other points in $\mathbb{P}^1$.

Now we could instead count the corresponding tropical objects, defining numbers $N_{\mathrm{trop}}(d,g)$, $E_{\mathrm{trop}}(d,j)$ and $H^g_{d,\mathrm{trop}}(\eta,\nu)$, and hope to end up with the same numbers. Each tropical object has to be counted with a certain tropical multiplicity that should reflect how many objects in the algebraic count degenerate to this tropical object. For the numbers $N(d,g)$, the Correspondence Theorem $N(d,g)=N_{\text{trop}}(d,g)$ has been shown in the pioneering work of Grigory Mikhalkin ([12]). The equality $E(d,j)=E_{\text{trop}}(d,j)$ was proved in [10] and $H^g_d(\eta,\nu)=H^g_{d,\text{trop}}(\eta,\nu)$ in [3].

The study of tropical enumerative numbers like the above requires an argument why the tropical count remains invariant under a deformation of the conditions, for instance, an argument why the numbers $N_{\text{trop}}(d,g)$ do not depend on the position of the $3d+g-1$ points (as long as they are in general position). The corresponding independence statements in algebraic geometry are a consequence of the fact that the enumerative numbers can be interpreted as intersection numbers of cycles on suitable moduli spaces, and that intersection products are invariant under deformation. In tropical geometry, we can construct analogues of the moduli spaces. Also, tropical intersection theory has been studied recently ([13],[1]). However, at this moment not all independence statements above can be deduced from general principles of tropical intersection theory. In fact, we can only use tropical intersection theory to prove independence statements for numbers of rational curves, that is, if the genus g is 0. For the numbers $N_{\text{trop}}(d,g)$, the independence was shown in [12] by relating the tropical numbers to classical ones for which the invariance is known. An alternative combinatorial proof determines the different possibilities of how a tropical curve can change when the points are deformed ([7]). For the numbers $E_{\text{trop}}(d,j)$ and $H^g_{d,\text{trop}}(\eta,\nu)$ the independence was shown in [10] and [3], respectively, using moduli space techniques. However, tedious case-by-case analyses and computations were necessary. The same techniques can also be used to prove the independence for the numbers $N_{\text{trop}}(d,g)$, and we outline this proof shortly in this survey since it cannot be found in the literature.

A main reason why we cannot prove the independence with tropical intersection theory is that the tropical moduli spaces we want to work with do not have a tropical structure yet. We can define them only as abstract weighted polyhedral complexes. The only case in which the tropical structure of the moduli space is well understood is for rational curves ([14], [5]). Since the corresponding moduli spaces in algebraic geometry are stacks rather than varieties, we expect that we need a rigorous definition of a tropical stack, which does not yet exist, before we can succeed in equipping the tropical moduli spaces with more structure. For genus 0, the notion of a tropical stack is avoided by introducing extra labelings that will remove automorphisms, see Section 7. Once we understand the tropical structure of the moduli spaces, we expect that tropical intersection theory should provide natural proofs of the independence statements, leading thus to a more rigorous set-up for tropical enumerative geometry.

The purpose of this survey is to describe the tropical moduli spaces used in the three enumerative problems mentioned above, and to observe similarities in their local structure. We hope a better understanding of tropical moduli spaces might eventually lead to a definition of a tropical structure for them.

We define three tropical moduli spaces that parametrize a larger set of objects than the ones we want to count. For the first enumerative problem (the numbers $N_{\text{trop}}(d,g)$) the space $\mathcal{M}_{g,n,\text{trop}}(\mathbb{R}^2,\Delta)$ parametrizes genus g, degree d plane tropical curves with $3d+g-1$ marked points. The space $\widetilde{\mathcal{M}}_{1,n,\text{trop}}(\mathbb{R}^2,\Delta)$ that we use for the second problem (the numbers $E_{\text{trop}}(d,j)$) parametrizes degree d elliptic plane tropical curves with $3d-1$ marked points. The space $\mathcal{M}'_{g,0,\text{trop}}(\mathbb{R}^1,\Delta)$ for the third problem (the numbers $H^g_{d,\text{trop}}(\eta,\nu)$) parametrizes degree d, genus g tropical maps to $\mathbb{P}^1$. Then we define maps from these moduli spaces, for instance, the map that evaluates the $3d+g-1$ marked points for the first problem. The inverse image under this map of a point configuration consists of those degree d genus g tropical curves that pass through the point configuration. Inverse image points have to be counted with the suitable tropical multiplicity. The map for the second enumerative problem evaluates $3d-1$ points and associates the tropical j-invariant. The map for the third problem evaluates the position of the vertices of the tropical curve that can be thought of as branch points.

We start in Section 2 with an example which should motivate our definition of tropical curves. In Section 3, we define abstract tropical curves and parametrized tropical curves. The latter can be thought of as analogues of stable maps. We also define spaces parametrizing tropical curves that can be thought of as analogues of moduli spaces of stable maps. We equip those tropical moduli spaces with the structure of a weighted polyhedral complex in Section 4. In Section 5, we define maps from the tropical moduli spaces that are used to define the tropical enumerative problems. We explain how our definition of $N_{\text{trop}}(d,g)$ relates to Mikhalkin's original definition of $N_{\text{trop}}(d,g)$ ([12]). In Section 6, we give a short overview of the independence proofs that have to be shown for each of the three enumerative problems. We give only short outlines of proofs. For more details, or for more formal definitions, see [10], [6] or [11].

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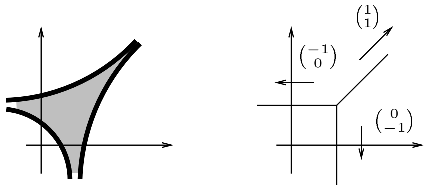
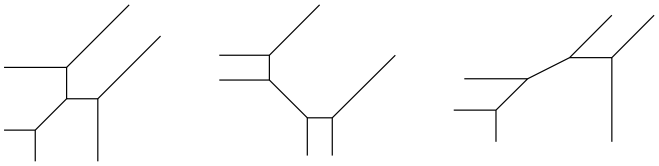
2. A MOTIVATING EXAMPLE

There are several ways to define the degeneration process which produces a tropical variety from an algebraic variety (see [12], [16], [4]). Here, we sketch just one basic example that motivates our later combinatorial definition of tropical curves. Let L be a projective line in $\mathbb{P}^2_{\mathbb{C}}$, and apply the map $\text{Log}\colon (\mathbb{C}^*)^2 \to \mathbb{R}^2$, $(s,t) \mapsto (\log|s|, \log|t|)$ to the restriction of L to $(\mathbb{C}^*)^2$. Let $(x:y:z)$ be the coordinates of $\mathbb{P}^2$, and identify $\mathbb{C}^2$ with the set $\{z \neq 0\}$. Then the map Log associates the point $(\log|\frac{x}{z}|, \log|\frac{y}{z}|) \in \mathbb{R}^2$ to a point $(x : y : z) \in \mathbb{P}^2$. The line L intersects the coordinate line $\{x = 0\}$ in one point. When we move along the line L towards the intersection with $\{x = 0\}$, the first coordinate of the image point under Log tends to $-\infty$. Also, when we move towards the intersection with the coordinate line $\{y = 0\}$, the second coordinate of the image tends to $-\infty$. When we move towards the intersection with $\{z = 0\}$, both coordinates become big and their difference tends to a constant. Furthermore, the image $\operatorname{Log}(L) \subset \mathbb{R}^2$ (called the amoeba of L) should be 2-dimensional, as the complex line has two real dimensions. These observations suggest that the image looks similar to the left of the following picture:



A tropical line can be thought of as a limit of this amoeba after shrinking it to something one-dimensional, as on the right in the picture above. (For more details on the limit process, see [12] or [4].) The only information kept are the three infinite rays and their directions. Note that the primitive integer vectors pointing in these three directions sum up to 0. This is called the *balancing condition* and is important in our combinatorial definition. Now let $C \subset \mathbb{P}^2$ be a conic. It intersects $\{x = 0\}$ in two points, $(0 : p_0 : 1)$ and $(0 : p_1 : 1)$. We can move along C near the first point and the first coordinate of the image will tend to $-\infty$, whereas the second tends to $\log |p_0|$. For the second point, the first coordinate will again tend to $-\infty$, but the second to $\log |p_1|$. Thus the amoeba of a conic has two "tentacles" in each of the three directions $(-1, 0)$, $(0, -1)$ and $(1, 1)$. We can not say precisely what happens in the middle. When we shrink the amoeba to something 1-dimensional to get an idea of how a tropical conic should look like, there are indeed several possibilities of what can happen in the middle.



The picture shows three different types of a tropical conic.

In many places in the literature, an alternate degeneration process is given; namely, take the image of the valuation map from an algebraic variety over the field of Puiseux series K (or another field with a non-archimedean valuation). Since this definition does not require taking a limit, it is more useful for computations ([2]). We define plane tropical curves combinatorially (Definition 3.6). For plane curves, it is true that any such combinatorial object (roughly, a graph satisfying the balancing condition) comes from an algebraic curve under the degeneration process ([17]).

Remark 2.1. We can also apply the degeneration to higher-dimensional varieties. In the case of constant coefficients (that is, if the ideal defining the variety is an ideal of $\mathbb{C}[\underline{x}] \subset K[\underline{x}]$) the image under the valuation map is a polyhedral fan that satisfies (a higher-dimensional version of) the balancing condition ([18], Section 2.5). The role of the primitive integer vector pointing in the direction of an edge is played by the lattice in a top-dimensional cone of the fan. Combinatorially, higher-dimensional tropical varieties are defined roughly as polyhedral complexes obtained by gluing fans that satisfy the balancing condition ([20], [5]). Not every such polyhedral complex comes from an algebraic variety under the degeneration process.

3. TROPICAL $M_{g,n}(\mathbb{P}^r, d)$

We want to define a tropical analogue of $M_{g,n}(\mathbb{P}^r, d)$; that is, we want to define maps from abstract tropical curves to $\mathbb{R}^r$ such that the images look like the tropical curves we have seen in Section 2 above (that is, like graphs satisfying the balancing condition). The abstract tropical curves should be marked by n points. We have seen above that the unbounded edges (or, ends) of a tropical curve can be thought of as coming from the intersection with coordinate hyperplanes. In this sense, the ends are special points of the tropical curve. Thus we define the tropical analogue of marked points to be marked ends.

Let us first fix some notation we want to use for graphs. Let Γ be a graph. Unbounded edges (also called *ends*) are allowed. We denote the set of vertices by Γ^0 and the set of edges Γ^1. The subset of ends is called Γ^1_∞ and the subset of bounded edges Γ^1_0. We call a pair $F = (V, e)$ where e is an edge of Γ and $V \in \partial e$ a *flag* of Γ and think of it as a "directed edge"—an edge pointing away from its end vertex V. The genus of a connected graph Γ is the first Betti number of Γ, $h_1(\Gamma, \mathbb{Z})$, that is, the number of independent cycles.

Definition 3.1. An *abstract tropical curve* is a connected graph Γ whose vertices have valence at least 3 and whose bounded edges e are equipped with a length $l(e) \in \mathbb{R}_{>0}$.

The *genus* of an abstract tropical curve is the genus of Γ.

An *abstract n-marked tropical curve* is a tuple $(\Gamma, x_1, \dots, x_n)$ where Γ is an abstract tropical curve and $x_1, \dots, x_n \in \Gamma^1_\infty$ are distinct ends of Γ.

The set of all n-marked tropical curves with exactly n ends and of genus g is called $M_{g,n,\text{trop}}$.

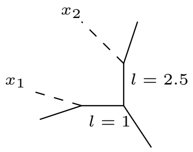
An *abstract tropical curve with labeled vertices* is an abstract tropical curve Γ where each vertex is labeled with $\operatorname{val}(V) - 2$ numbers such that the disjoint union of all vertex labelings equals $\{1, \ldots, s-2+2g\}$, where s is the number of ends and g is the genus.

For $i \in \{1, \ldots, s-2+2g\}$ we denote by V_i the vertex which has the label i. Note that for a curve with higher-valent vertices it is possible that $V_i = V_j$ for $i \neq j$ in this notation.

The *combinatorial type* α of an abstract tropical curve is the information left when dropping the lengths of the bounded edges.

Remark 3.2. We need vertex labelings only for the third enumerative problem, since tropical branch points of a map to tropical $\mathbb{P}^1$ are thought of as vertices of the underlying abstract tropical curve. Therefore we have to define two types of moduli spaces, one parametrizing curves with vertex labelings (but without marked ends) and one with marked ends.

Example 3.3. The following picture shows a 2-marked rational abstract tropical curve (without labeled vertices). Marked ends are drawn as dotted lines.



Remark 3.4. A connected graph of genus g has $\#\Gamma_0^1 = \#\Gamma_\infty^1 - 3 + 3g - \sum_V (\operatorname{val} V - 3)$ bounded edges. In particular, a 3-valent graph has $\#\Gamma_0^1 = \#\Gamma_\infty^1 - 3 + 3g$ bounded edges. A 3-valent graph of genus g has $\#\Gamma^0 = \#\Gamma_\infty^1 - 2 + 2g$ vertices. We need these relations for dimension counts later on.

Remark 3.5. For rational curves, the space $M_{0,n,\text{trop}}$ is known to be the space of trees, or a quotient of the tropical Grassmanian ([19], [14], [5]). It is in fact equal to the tropicalization of $M_{0,n}$ (where $M_{0,n}$ is realized as a quotient of the Grassmanian) ([8], proposition 5.8). Therefore it is a fan satisfying the balancing condition as mentioned in remark 2.1.

Definition 3.6. A *(parametrized) tropical curve in* $\mathbb{R}^r$ *(with labeled vertices)* is a tuple (Γ, h) where Γ is an abstract tropical curve (with labeled vertices) and $h\colon \Gamma \to \mathbb{R}^r$ is a continuous map satisfying:

(1) h maps each edge e of length $l(e)$ affinely to a line segment with rational slope in $\mathbb{R}^r$, that is, if we identify the edge e with the interval $[0, l(e)]$ (or $[0, \infty)$ for an end), h is of the form

$$h(t) = a + t \cdot v$$

for some $a \in \mathbb{R}^2$ and $v \in \mathbb{Z}^2$. The integral vector v occurring in this equation if $V \in \partial e$ is identified with 0 will be denoted $v(V, e)$ and called the *direction* of the flag (V, e). For an end e, we call its direction $v(e) = v(V, e)$ (where V is its only end vertex).

(2) At every vertex $V \in \Gamma^0$, the *balancing condition* is fullfilled:

$$\sum_{e \mid V \in \partial e} v(V,e) = 0.$$

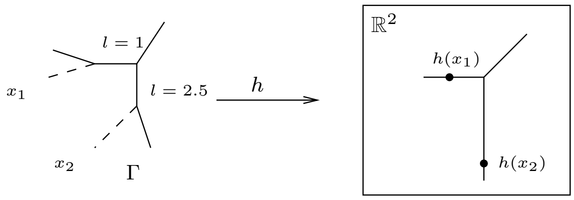
Note that $v(V,e) = -v(V',e)$ if $\{V,V'\} = \partial e$.

Definition 3.7. An *n-marked (parametrized) tropical curve in* $\mathbb{R}^r$ is a tuple $(\Gamma, h, x_1, \dots, x_n)$ where (Γ, h) is a tropical curve in $\mathbb{R}^r$, and $x_1, \dots, x_n \in \Gamma^1_\infty$ are distinct ends of Γ that are mapped to a point in $\mathbb{R}^2$ by h (that is, $v(x_i) = 0$).

Definition 3.8.

(1) The *genus* of a tropical curve in $\mathbb{R}^r$ is the genus of the underlying abstract tropical curve.
(2) The *combinatorial type* of a tropical curve in $\mathbb{R}^r$ is given by the data of the combinatorial type of the underlying abstract tropical curve Γ together with the directions of all its edges (bounded edges as well as ends).
(3) The *degree* of a tropical curve in $\mathbb{R}^r$ is defined to be the multiset $\Delta = \{v(e);\ e \in \Gamma^1_\infty \backslash \{x_1, \dots, x_n\}\}$ of directions of its ends. If this degree consists of the vectors $-e_0, -e_1, \dots, -e_r$, where $e_0 := -e_1 - \cdots - e_r$, and where each vector appears d times, we say that these curves have degree d.

Example 3.9. The following picture shows a rational tropical curve of degree 1 in $\mathbb{R}^2$ with two marked points.



Remark 3.10. Note that the direction vector $v(V,e)$ of a flag (V,e) (if it is nonzero) can uniquely be written as a product of a positive integer (called the weight of the edge e) and a primitive integer vector.

Remark 3.11. The map h of a tropical curve $(\Gamma, h, x_1, \dots, x_n)$ does not need to be injective on the edges. It is allowed that $v(V,e) = 0$ for a flag (V,e), that is, the edge e is contracted to a point in $\mathbb{R}^2$. The remaining flags around the vertex V then satisfy the balancing condition. If V is a 3-valent vertex, this means that the two other flags (V,e_1) and (V,e_2) around V have to satisfy $v(V,e_1) = -v(V,e_2)$, that is, they point in opposite directions. Hence, the image $h(\Gamma)$ looks locally around $h(V)$ like a straight line.

This holds in particular for the marked ends $x_1, \dots, x_n$, as they are required to be mapped to a point. Therefore, they can be seen as tropical analogues of the marked points of stable maps.

Note that the contracted bounded edges also lead to "hidden moduli parameters": if we vary the length of a contracted bounded edge, then we arrive at a family of different parametrized tropical curves whose images in $\mathbb{R}^2$ are all the same.

We are now ready to define the two types of moduli spaces mentioned in Remark 3.2.

Definition 3.12. For all g, $n \geq 0$ and Δ, let $M_{g,n,\text{trop}}(\mathbb{R}^2, \Delta)$ be the set of all n-marked tropical curves $(\Gamma, h, x_1, \ldots, x_n)$ in $\mathbb{R}^2$ of degree Δ and genus $g' \leq g$.

For all $g \geq 0$ and Δ, let $M'_{g,0,\text{trop}}(\mathbb{R}^1, \Delta)$ be the set of all tropical curves with labeled vertices (Γ, h) in $\mathbb{R}^1$ of degree Δ and genus $g' \leq g$.

We denote by $M^{\alpha}_{g,n,\text{trop}}(\mathbb{R}^2, \Delta)$ and $M'^{\alpha}_{g,0,\text{trop}}(\mathbb{R}^1, \Delta)$, respectively, the subsets of $M_{g,n,\text{trop}}(\mathbb{R}^2, \Delta)$ and $M'_{g,0,\text{trop}}(\mathbb{R}^1, \Delta)$ of tropical curves of combinatorial type α.

We need to include curves of lower genus here, since they appear in the boundary of types of genus g.

There are only finitely many combinatorial types in $M_{g,n,\text{trop}}(\mathbb{R}^2, \Delta)$ and $M'_{g,0,\text{trop}}(\mathbb{R}^1, \Delta)$ ([6]).

Lemma 3.13. *The subsets $M^{\alpha}_{g,n,\text{trop}}(\mathbb{R}^2, \Delta)$ and $M'^{\alpha}_{g,0,\text{trop}}(\mathbb{R}^1, \Delta)$ are unbounded open convex polyhedra in real vector spaces of dimension $2 + \#\Gamma^1_0$ and $1 + \#\Gamma^1_0$, respectively. They have, respectively, two or one coordinates $h(V)$ for the position of a root vertex V and coordinates $l(e)$ for the lengths of all bounded edges e. They are cut out by the inequalities that all lengths have to be positive and by the equations for the loops. If Γ is 3-valent, the expected dimensions are $2 + \#\Gamma^1_0 - 2g = \#\Delta - 1 + g$ for $M^{\alpha}_{g,n,\text{trop}}(\mathbb{R}^2, \Delta)$, and $1 + \#\Gamma^1_0 - g = \#\Delta - 2 + 2g$ for $M'^{\alpha}_{g,0,\text{trop}}(\mathbb{R}^1, \Delta)$.*

Proof. Given a curve of type α we can recover the map h from the data of the position of one root vertex. This is true because the directions are fixed by α and the lengths are fixed by the abstract curve. Thus $M^{\alpha}_{g,n,\text{trop}}(\mathbb{R}^2, \Delta)$ and $M'^{\alpha}_{g,0,\text{trop}}(\mathbb{R}^1, \Delta)$ are parametrized by the position $h(V_1)$ and the lengths of all bounded edges. The length coordinates have to satisfy the conditions that the g loops close up in the image in $\mathbb{R}^2$ (resp. $\mathbb{R}$). Each loop gives two (resp. one) conditions, but they do not have to be linearly independent. The statement about the expected dimension follows from Remark 3.4. □

A different choice of the root vertex or of the order of the bounded edges leads to a linear isomorphism on $M^{\alpha}_{g,n,\text{trop}}(\mathbb{R}^2, \Delta)$ (resp. $M'^{\alpha}_{g,0,\text{trop}}(\mathbb{R}^1, \Delta)$) of determinant ± 1. This is obvious for the order of the bounded edges. If we choose another root vertex V', the difference $h(V) - h(V')$ of the images of the two vertices is given by $\sum_{(W,e)} l(e) \cdot v(W, e)$, where the sum is taken over a chain of flags leading from V to V'. This is obviously a linear combination of the lengths of the bounded edges. As these length coordinates themselves remain unchanged it is clear that the determinant of this change of coordinates is 1.

For any type α, the boundary of an open polyhedron $M^{\alpha}_{g,n,\mathrm{trop}}(\mathbb{R}^2,\Delta)$ or $M'^{\alpha}_{g,0,\mathrm{trop}}(\mathbb{R}^1,\Delta)$ consists of curves where some length coordinates are shrunk to 0. We can remove those edges and obtain a new tropical curve of a combinatorial type α', possibly of lower genus. The following picture shows how this can look like locally. The edges which tend to have length zero when we move towards the boundary are drawn in bold.

Thus we can glue the spaces $M^{\alpha}_{g,n,\mathrm{trop}}(\mathbb{R}^2,\Delta)$ or $M'^{\alpha}_{g,0,\mathrm{trop}}(\mathbb{R}^1,\Delta)$ along their boundaries.

4. The moduli spaces as weighted polyhedral complexes

Definition 4.1. Let $X_1,\ldots,X_N$ be (possibly unbounded) open convex polyhedra in real vector spaces. A *polyhedral complex* with cells $X_1,\ldots,X_N$ is a topological space X together with continuous inclusion maps $i_k\colon \overline{X_k}\to X$ such that X is the disjoint union of the sets $i_k(X_k)$ and the coordinate change maps $i_k^{-1}\circ i_l$ are affine (where defined) for all $k\neq l$. We usually drop the inclusion maps i_k in the notation and say that the cells X_k are contained in X.

The *dimension* $\dim X$ of a polyhedral complex X is the maximum of the dimensions of its cells. We say that X is of *pure dimension* $\dim X$ if every cell is contained in the closure of a cell of dimension $\dim X$. A point of X is said to be *in general position* if it is contained in a cell of dimension $\dim X$. For a point P in general position, we denote the cell of dimension $\dim X$ in which it is contained by X_P.

A *weighted polyhedral complex* is a polyhedral complex such that there is a weight $w(X_i)\in\mathbb{Q}$ associated to each cell X_i of highest dimension.

Now we want to glue the polyhedra $M^{\alpha}_{g,n,\mathrm{trop}}(\mathbb{R}^2,\Delta)$ or $M'^{\alpha}_{g,0,\mathrm{trop}}(\mathbb{R}^1,\Delta)$ to a weighted polyhedral complex. However, we want the polyhedral complex to be of the expected dimension, so in each case, we have to throw away certain strata. Later on we define maps from the moduli space to some other space that we use to impose conditions. We count tropical curves in the inverse image of a point. The strata whose dimension is too high are not mapped injectively and therefore do not contribute to the count. Thus we can drop them.

Also, we have to define weights for the top-dimensional strata. For this, we need the following definitions:

Definition 4.2. We call a type α in $M_{g,n,\mathrm{trop}}(\mathbb{R}^2,\Delta)$ or $M'_{g,0,\mathrm{trop}}(\mathbb{R}^1,\Delta)$ *regular* if the underlying graph is 3-valent and of genus g, and the g loops impose independent conditions.

Definition 4.3. Let α be a regular combinatorial type in $M_{g,n,\mathrm{trop}}(\mathbb{R}^2,\Delta)$ (resp. $M'_{g,0,\mathrm{trop}}(\mathbb{R}^1,\Delta)$). Pick g independent cycles of Γ, that is, generators of $H_1(\Gamma,\mathbb{Z})$. Each such generator is given as a chain of flags around the loop. Define a $2g \times 2+\#\Gamma_0^1 = 2g\times n+\#\Delta-1+3g$ (resp. $g\times 1+\#\Gamma_0^1 = g\times\#\Delta-2+3g$) matrix A_α with two (one) columns for the position of a root vertex $h(V)$ and a column for each length coordinate, and with two (resp. one) rows for each cycle containing the equation of the loop in $\mathbb{R}^2$ (resp. $\mathbb{R}$) (depending on the lengths of the bounded edges in the loop):

$$\sum_{(W,e)} v(W,e)\cdot l(e),$$

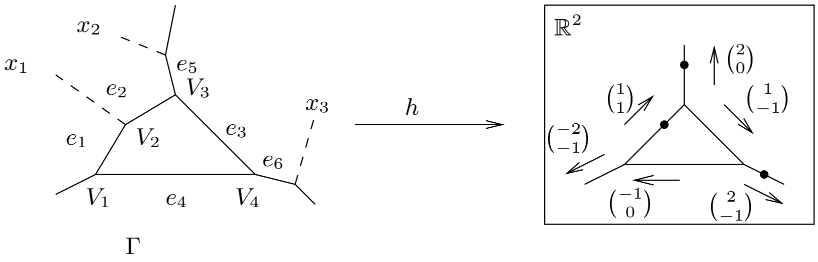
where the sum now goes over the chosen chain of flags around the loop. Then $A_\alpha\colon \mathbb{R}^{n+\#\Delta-1+3g}\to\mathbb{R}^{2g}$ (resp. $A_\alpha\colon \mathbb{R}^{\#\Delta-2+3g}\to\mathbb{R}^{g}$) is a linear map.

Denote by I_α the index of the sublattice $A_\alpha(\mathbb{Z}^{n+\#\Delta-1+3g})\subset\mathbb{Z}^{2g}$ (resp. $A_\alpha(\mathbb{Z}^{\#\Delta-2+3g})\subset\mathbb{Z}^{g}$).

Note that $M^\alpha_{g,n,\mathrm{trop}}(\mathbb{R}^2,\Delta)$ (resp. $M'^\alpha_{g,0,\mathrm{trop}}(\mathbb{R}^1,\Delta)$) equals the intersection of $\mathbb{R}^2\times(\mathbb{R}_{>0})^{\#\Gamma_0^1}$ (resp. $\mathbb{R}\times(\mathbb{R}_{>0})^{\#\Gamma_0^1}$) with the kernel of this map. This is true because we force the images of the cycles in $\mathbb{R}^2$ (resp. $\mathbb{R}$) to close up by requiring that the equations of the chains of flags are 0.

Note also that I_α does not depend on the chosen generators of $H_1(\Gamma,\mathbb{Z})$: If we choose another set of generators, these new generators are given as linear combinations with coefficients in $\mathbb{Z}$ of the old generators, so the rowspace of the matrix is not changed.

Example 4.4. The picture shows a regular curve C in $\mathcal{M}_{1,3,\mathrm{trop}}(\mathbb{R}^2,\Delta)$ (where $\Delta=\{(-2,-1),(0,2),(2,-1)\}$).



Choose the chain of flags $(V_1, e_1), \dots, (V_4, e_4)$ around the cycle. The directions of those four flags are $(1, 1)$, $(1, 1)$, $(1, -1)$ and $(-1, 0)$. Thus the map $A_\alpha \colon \mathbb{R}^8 \to \mathbb{R}^2$ is given by the following matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

where the coordinates of $\mathbb{R}^8$ are $h(V_1)$, $l(e_1), \dots, l(e_6)$.

4.1. The moduli space $\mathcal{M}_{g,n,\mathrm{trop}}(\mathbb{R}^2, \Delta)$ for the first enumerative problem, the numbers $N_{\mathrm{trop}}(d, g)$.

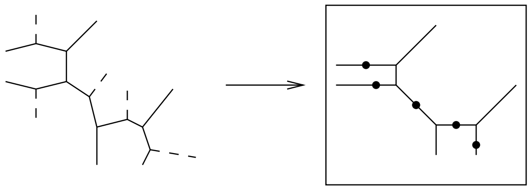
Definition 4.5. Let $C = (\Gamma, h, x_1, \dots, x_n)$ be a tropical curve. If C has no contracted bounded edges (that is, no direction vector $v(e) = 0$ for $e \in \Gamma_0^1$), and if for all V such that there are two adjacent flags of the same direction $v(V, e_1) = v(V, e_2)$ the directions of the flags adjacent to V span $\mathbb{R}^2$, then C is called *relevant*. (In particular, every such vertex is at least 4-valent.)

We define $\mathcal{M}_{g,n,\mathrm{trop}}(\mathbb{R}^2, \Delta)$ to be the subset of $M_{g,n,\mathrm{trop}}(\mathbb{R}^2, \Delta)$ of relevant tropical curves which satisfy in addition the following property: if they are of genus $g' < g$, then they appear in the boundary of a relevant type of genus g.

The weight $w_1(\alpha)$ of a top-dimensional cell $M^\alpha_{g,n,\mathrm{trop}}(\mathbb{R}^2, \Delta)$ is defined to be the index I_α from Definition 4.3.

It follows from Proposition 4.1 in [11] that all types of top dimension in $\mathcal{M}_{g,n,\mathrm{trop}}(\mathbb{R}^2, \Delta)$ are regular. Thus the weight is well-defined.

The following picture shows an element of $\mathcal{M}_{0,5,\mathrm{trop}}(\mathbb{R}^2, 2)$:



4.2. The moduli space $\widetilde{\mathcal{M}}_{1,n,\mathrm{trop}}(\mathbb{R}^2, \Delta)$ for the second enumerative problem, the numbers $E_{\mathrm{trop}}(d, j)$.

Let α be a combinatorial type in $M_{1,n,\mathrm{trop}}(\mathbb{R}^2, \Delta)$. The *deficiency* $\operatorname{def}(\alpha)$ is defined to be

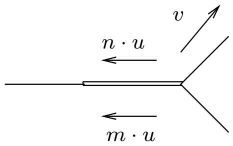
$$\operatorname{def}(\alpha) = \begin{cases} 2 \text{ if } g = 1 \text{ and the cycle is mapped to a point in } \mathbb{R}^2, \\ 1 \text{ if } g = 1 \text{ and the cycle is mapped to a line in } \mathbb{R}^2, \\ 0 \text{ otherwise.} \end{cases}$$

Since the loop imposes either two, one or no condition (depending on whether it spans $\mathbb{R}^2$, is mapped to a line or to a point), we can determine the dimension of $M^\alpha_{1,n,\mathrm{trop}}(\mathbb{R}^2, \Delta)$ exactly to be $\#\Delta + n + g - 1 - \sum_V (\operatorname{val} V - 3) + \operatorname{def}(\alpha)$ ([10]).

Definition 4.6. Remove from $M_{1,n,\text{trop}}(\mathbb{R}^2,\Delta)$ the cells of dimension bigger than $\#\Delta+n$ and cells of rational curves which are not contained in the boundary of a cell corresponding to a genus 1 curve. The remaining subset of $M_{1,n,\text{trop}}(\mathbb{R}^2,\Delta)$ is called $\widetilde{\mathcal{M}}_{1,n,\text{trop}}(\mathbb{R}^2,\Delta)$. We associate the following weights to the strata of dimension $\#\Delta+n$:

(1) Assume $\operatorname{def}(\alpha)=0$, and the curves of type α are of genus 1. Then we associate the weight $w_2(\alpha)=I_\alpha\cdot(\frac{1}{2})^r$, where r denotes the number of vertices V such that $\Gamma\setminus V$ has two connected components of the same combinatorial type (that is, for which both the abstract graph and the directions coincide).

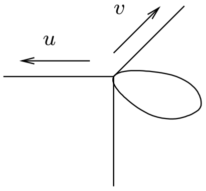
(2) Assume $\operatorname{def}(\alpha)=1$. By the dimension count there is a 4-valent vertex. Assume first that the 4-valent vertex is adjacent to the cycle, that is, locally the curves look like the following picture:



In the notations above, $n\cdot u$, $m\cdot u$ and v denote the direction vectors of the corresponding edges (n and m are chosen such that their greatest common divisor is 1). If $n\neq m$, or if $n=m=1$ and the cycle is formed by three edges due to the presence of a marked point, we associate the weight $w_2(\alpha)=|\det(u,v)|$. If $n=m=1$ and no point is on the flat cycle, then we associate $w_2(\alpha)=\frac{1}{2}|\det(u,v)|$. (Due to the balancing condition this definition is not dependent of the choice of v.)

In case the 4-valent vertex is not adjacent to the cycle, we associate the weight 0.

(3) Assume $\operatorname{def}(\alpha)=2$. Assume first that the 5-valent vertex is adjacent to the cycle, that is, locally the curves look like this:

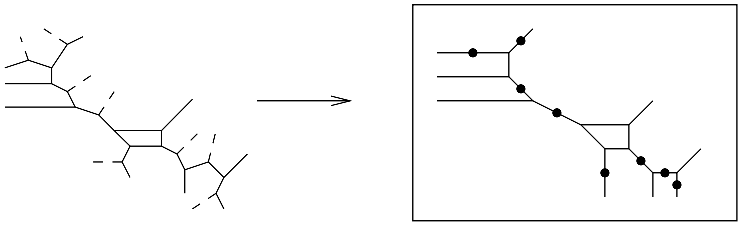


where u and v denote the direction vectors of the corresponding edges. We associate the weight $w_2(\alpha)=\frac{1}{2}(|\det(u,v)|-1)$. (Note that due to the balancing condition this definition is independent of the choice of u and v.) In the case that there are two 4-valent vertices or that the 5-valent vertex is not adjacent to the cycle, we associate the weight 0.

The factor of $(\frac{1}{2})^r$ was left out in the original definition of $\widetilde{\mathcal{M}}_{1,n,\text{trop}}(\mathbb{R}^2,\Delta)$ in [10]. The reason is that curves with vertices V such that $\Gamma \setminus V$ has two connected components of the same type count with multiplicity 0 later, since they are not mapped injectively.

Remark 4.7. Note that we include types in $\widetilde{\mathcal{M}}_{1,n,\text{trop}}(\mathbb{R}^2,\Delta)$ which are not relevant and thus not included in $\mathcal{M}_{g,n,\text{trop}}(\mathbb{R}^2,\Delta)$. The reason is that the map we want to use for the first enumerative problem only evaluates at different points, whereas the map for the second enumerative problem takes the cycle length of the tropical curve into account (see Definition 5.5). A cell corresponding to a non-relevant type like the one with $\operatorname{def}(\alpha) = 1$ above is not mapped injectively with just evaluations, because we can change the length coordinates in the cycle without changing the position of any marked point. It is mapped injectively in the second problem though, because a change of length coordinates in the cycle changes the cycle length. Therefore we have to consider it in the second problem, but not in the first one.

The following picture shows an element of $\widetilde{\mathcal{M}}_{1,8,\text{trop}}(\mathbb{R}^2,3)$:



4.3. The moduli space $\mathcal{M}'_{g,0,\text{trop}}(\mathbb{R}^1,\Delta)$ for the third enumerative problem, the numbers $H^g_{d,\text{trop}}(\eta,\nu)$.

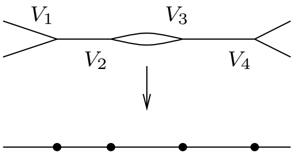
As mentioned in Remark 3.2, we need vertex labels here.

Definition 4.8. Let $\mathcal{M}'_{g,0,\text{trop}}(\mathbb{R}^1,\Delta)$ be the subset of $M'_{g,0,\text{trop}}(\mathbb{R}^1,\Delta)$ containing all combinatorial types α such that if $M'^{\alpha}_{g,0,\text{trop}}(\mathbb{R}^1,\Delta)$ is of dimension $\#\Delta - 2 + 2g$ or bigger then α is regular and if $M'^{\alpha}_{g,0,\text{trop}}(\mathbb{R}^1,\Delta)$ is of dimension less than $\#\Delta - 2 + 2g$ then it is contained in a cell corresponding to a regular type.

In particular, the top dimension of $\mathcal{M}'_{g,0,\text{trop}}(\mathbb{R}^1,\Delta)$ is $\#\Delta - 2 + 2g$. We define the weight $w_3(\alpha)$ of a top-dimensional cell as the product of three types of factors:

- the index I_α;
- $\frac{1}{2}$ for every vertex V such that $\Gamma \setminus V$ has two connected components of the same combinatorial type;
- $\frac{1}{2}$ for every cycle which consists of two edges which have the same direction.

The following picture shows an element of $\mathcal{M}'_{1,0,\text{trop}}(\mathbb{R}^1,2)$:



It is easy to show now that the three moduli spaces are indeed weighted polyhedral complexes.

Remark 4.9. Note that the weights in Definitions 4.5, 4.6 and 4.8 coincide. We do not need factors of $\frac{1}{2}$ in Definition 4.5 because a regular and relevant curve cannot have a vertex V such that $\Gamma \setminus V$ has two connected components of the same combinatorial type or a cycle consisting of two edges which have the same weight. Also, we do not need the special cases (2) and (3) of 4.6 in either of the two other definitions. They are not relevant. In the third enumerative problem, they are not of top dimension.

The factors of $\frac{1}{2}$ can be thought of as taking care of automorphisms (see also Section 7).

5. THE TROPICAL ENUMERATIVE PROBLEMS

Definition 5.1. A *morphism* between a weighted polyhedral complex X and a polyhedral complex Y is a continuous map $f\colon X \to Y$ such that for each cell $X_i \subset X$ the image $f(X_i)$ is contained in only one cell of Y, and $f|_{X_i}$ is a linear map (of polyhedra).

Assume $f\colon X \to Y$ is a morphism of weighted polyhedral complexes of the same pure dimension, and $P \in X$ is a point such that both P and $f(P)$ are in general position (in X and Y, respectively). Then locally around P the map f is a linear map between vector spaces of the same dimension. We denote by D_P the absolute value of the determinant of this linear map and define the *multiplicity* $\operatorname{mult}_f(P) = D_P \cdot w(X_P)$ of f at P to be D_P times the weight of the cell X_P, $w(X_P)$. Note that the multiplicity depends only on the cell X_P of X in which P lies. We call it the multiplicity of f in this cell.

A point $Q \in Y$ is said to be *in f-general position* if Q is in general position in Y and all points of $f^{-1}(Q)$ are in general position in X. Note that the set of points in f-general position in Y is the complement of a subset of Y of dimension at most $\dim Y - 1$; in particular it is a dense open subset. Now if $Q \in Y$ is a point in f-general position we define the *degree* of f at Q to be

$$\deg_f(Q) := \sum_{P \in f^{-1}(Q)} \operatorname{mult}_f(P).$$

Note that this sum is indeed finite: first of all there are only finitely many cells in X. Moreover, in each cell (of maximal dimension) of X where f is not injective (that is, where there might be infinitely many inverse image points of Q) the determinant of f is zero and hence so is the multiplicity for all points in this cell.

Moreover, since X and Y are of the same pure dimension, the cells of X on which f is not injective are mapped to a locus of codimension at least 1 in Y. Thus the set of points in f-general position away from this locus is also a dense open subset of Y, and for all points in this locus we have that not only the sum above but indeed the fiber of Q is finite.

Note that the definition of multiplicity $\operatorname{mult}_f(P)$ in general depends on the coordinates we choose for the cells. However, we will use this definition only for morphisms for which D_P, the absolute value of the determinant, does not depend on the chosen coordinates, if they are chosen in a natural way; in our case this means we choose lattice bases of the spaces $M^{\alpha}_{g,n,\mathrm{trop}}(\mathbb{R}^2,\Delta)$ and $M'^{\alpha}_{g,0,\mathrm{trop}}(\mathbb{R}^1,\Delta)$. Choosing a different lattice basis leads to a base change matrix of determinant ±1 which does not change the multiplicity. Since D_P depends only on the cell and for us cells correspond to combinatorial types α, we will use the notation D_α.

As lattice bases are in general hard to compute, we use the following easier way to determine $\operatorname{mult}_f(C)$ for a morphism starting from one of our moduli spaces:

Construction 5.2. Let $f\colon \mathcal{M} \to Y$ be a morphism of weighted polyhedral complexes of the same pure dimension, where $\mathcal{M}$ is $\mathcal{M}_{g,n,\mathrm{trop}}(\mathbb{R}^2,\Delta)$, $\widetilde{\mathcal{M}}_{1,n,\mathrm{trop}}(\mathbb{R}^2,\Delta)$ or $\mathcal{M}'_{g,0,\mathrm{trop}}(\mathbb{R}^1,\Delta)$. For a regular type α the cell $M = M^{\alpha}_{g,n,\mathrm{trop}}(\mathbb{R}^2,\Delta)$ (resp. $M = M'^{\alpha}_{g,0,\mathrm{trop}}(\mathbb{R}^1,\Delta)$) is cut out of $V = \mathbb{R}^{2+\#\Gamma^1_0}$ (resp. $V = \mathbb{R}^{1+\#\Gamma^1_0}$) by the inequalities that lengths are positive and by $d = 2g$ (resp. $d = g$) independent equations for the loops. Pick a map $\tilde{f}_\alpha\colon V \to Y \times \mathbb{R}^d$ such that $\tilde{f}_\alpha|_M = f|_M \times A_\alpha$, where A_α is the map containing the equations for the loops as in 4.3.

Lemma 5.3. *For a map $\tilde{f}_\alpha$ (defined for a regular type α) from Construction* 5.2 *we have $|\det(\tilde{f}_\alpha)| = I_\alpha \cdot D_\alpha = \operatorname{mult}_f(C)$, where C is a curve of type α. In particular $|\det(\tilde{f}_\alpha)|$ does not depend on the choice of $\tilde{f}_\alpha$.*

This is basically a straightforward lattice index computation, which can be found in [15], Lemma 1.6. The index of a square integer matrix is just the absolute value of its determinant, and the index of a product of two maps $f \times g$ is equal to the index of $f|_{\ker g}$ times the index of g. Remember that $\tilde{f}_\alpha = f|_M \times A_\alpha$ and M (that is, the cell of type α) is the kernel of the map A_α (intersected with the conditions that lengths have to be positive) by Definition 4.3.

Example 5.4. For the maps we will use, we can choose a possible $\tilde{f}_\alpha$ just by choosing chains of flags to the marked points, respectively to the vertices. For the curve C in Example 4.4, choose V_1 to be the root vertex, and go from V_1 to x_1 via (V_1, e_1). Go to x_2 from V_1 via (V_1, e_1), (V_2, e_2) and (V_3, e_5) and to x_3 via (V_1, e_1), (V_2, e_2), (V_3, e_3) and (V_4, e_6). Thus $\tilde{f}_\alpha$ is the $2 + \#\Gamma_0^1 = 8$ times $2n + 2 = 8$ matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

where the coordinates of $\mathbb{R}^8$ are $h(V_1)$, $l(e_1), \ldots, l(e_6)$. Note that we could for example also have gone to x_3 via (V_1, e_4) in which case the fifth and sixth row would be replaced by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

This matrix differs from the other only by subtracting the seventh from the fifth and the eighth from the sixth line—that is, we subtract the two equations for the loop from one chain of flags to get to the other chain of flags. In particular choosing a different chain of flags does not change the absolute value of the determinant. We will see that the map $\tilde{f}_\alpha$ we describe here equals $\mathrm{ev}|_M \times A_\alpha$ when restricted to the cell $M = M^\alpha_{1,3,\mathrm{trop}}(\mathbb{R}^2, \Delta)$ (the map ev is defined in 5.5). Then by Lemma 5.3 we have $2 = |\det(\tilde{f}_\alpha)| = \mathrm{mult}_{\mathrm{ev}}(C)$.

Definition 5.5. Let

$$\mathrm{ev}_i \colon M_{g,n,\mathrm{trop}}(\mathbb{R}^2, \Delta) \to \mathbb{R}^2,\ (\Gamma, h, x_1, \ldots, x_n) \mapsto h(x_i)$$

denote the i-th evaluation map. By $\mathrm{ev} = \mathrm{ev}_1 \times \ldots \times \mathrm{ev}_n$ we denote the combination of all n evaluation maps.

The j-invariant of an elliptic curve tropicalizes to the cycle length ([9]). For a tropical curve $C = (\Gamma, h, x_1, \ldots, x_n)$ of genus 1, we pick a generator of $H_1(\Gamma, \mathbb{Z})$ given as a chain of flags. If we avoid passing any edge in two directions, it is unique up to orientation. We define the cycle length to be the sum of the lengths of the edges which are part of this cycle. This can also be expressed in terms of forgetful maps ([10]). We define a map $j \colon \widetilde{\mathcal{M}}_{1,n,\mathrm{trop}}(\mathbb{R}^2, \Delta) \to \mathbb{R}_{\geq 0}$ sending C to its cycle length. For a rational tropical curve, we say $j(C) = 0$. Define

$$\pi := \mathrm{ev} \times j \colon \widetilde{\mathcal{M}}_{1,n,\mathrm{trop}}(\mathbb{R}^2, \Delta) \to \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0}.$$

We define the tropical branch map δ as

$$\delta \colon \mathcal{M}'_{g,0,\mathrm{trop}}(\mathbb{R}^1, \Delta) \to \mathbb{R}^{\#\Delta - 2 + 2g} \colon (\Gamma, h) \mapsto (h(V_1), \ldots, h(V_{\#\Delta - 2 + 2g})),$$

where V_i is the vertex in Γ^0 with label i as defined in 3.1.

All those maps are morphisms of weighted polyhedral complexes. For example, the position $h(x_i)$ equals $h(V) + \sum v(V,e) \cdot l(e)$ where the sum goes over a chain of flags leading from V to x_i. This expression is linear in the coordinates $h(V)$ and $l(e)$.

Now we define the tropical enumerative numbers.

Definition 5.6.

(1) Let $n = \#\Delta + g - 1$. For a point configuration $\mathcal{P} \in \mathbb{R}^{2n}$ in ev-general position, define $N_{\text{trop}}(\Delta, g) = \deg_{\text{ev}}(\mathcal{P})$.
(2) Let $n = \#\Delta - 1$. For a point configuration $\mathcal{P} \in \mathbb{R}^{2n}$ in π-general position, define $E_{\text{trop}}(\Delta, j) = \deg_{\pi}(\mathcal{P})$.
(3) For a point configuration $\mathcal{P} \in \mathbb{R}^{\#\Delta - 2 + 2g}$ in δ-general position, define $H^g_{d,\text{trop}}(\eta, \nu) = \deg_{\delta}(\mathcal{P})$.

The question as posed in the introduction is now why those numbers do not depend on the point $\mathcal{P}$, that is, why the degrees of the three maps are constant. We give an outline of these proofs in Section 6.

Note that $N_{\text{trop}}(\Delta, g)$ is defined differently in [12]: the tropical curves there are counted with a multiplicity $\text{mult}(C)$ which is not defined via the evaluation map. We show that $\text{mult}(C)$ for a relevant and regular curve C of type α coincides with $\text{mult}_{\text{ev}}(C)$ ([11]).

Definition 5.7. The multiplicity of a 3-valent vertex V is defined to be the absolute value of the determinant $\det(v_1, v_2)$, where v_1 and v_2 are two directions of flags adjacent to V. The balancing condition tells us that it makes no difference which two of the three flags adjacent to V we choose. The *multiplicity* $\text{mult}(C)$ of a 3-valent tropical curve is defined to be the product of the multiplicities of all vertices ([12]).

Example 5.8. The multiplicity of the curve C from Example 4.4 equals $\text{mult}(C) = 2$. As we have seen in Example 5.4, $\text{mult}_{\text{ev}}(C) = 2$, too.

A *string* in C is a subgraph of Γ homeomorphic either to $\mathbb{R}$ or to S^1 (that is, a "path" starting and ending with an unbounded edge, or a path around a loop) that does not intersect the closures $\overline{x_i}$ of the marked points.

Definition 5.9. For a tropical curve C of regular type α, pick a chain of flags for each marked point x_i leading from the root vertex V to x_i. Define a matrix $\widetilde{\text{ev}}_\alpha \colon \mathbb{R}^{2+\#\Gamma^1_0} \to \mathbb{R}^{2n} \times \mathbb{R}^{2g}$ with two rows for each marked point containing the chain of flags and two rows for each loop containing the equation of the loop (as in Definition 4.3).

Remark 5.10. Note that $\widetilde{\text{ev}}_\alpha = \text{ev} \times A_\alpha$ on $M^{\alpha}_{g,n,\text{trop}}(\mathbb{R}^2, \Delta)$, where A_α is defined in 4.3 and $M^{\alpha}_{g,n,\text{trop}}(\mathbb{R}^2, \Delta)$ is the kernel of A_α intersected with the conditions that the lengths are positive. The map $\widetilde{\text{ev}}_\alpha$ depends on the choices, but $|\det \widetilde{\text{ev}}_\alpha|$ does not since $|\det \widetilde{\text{ev}}_\alpha| = w(\alpha) \cdot D_\alpha = \text{mult}_{\text{ev}}(C)$

by Lemma 5.3. By abuse of notation, we still speak of *the map* $\widetilde{\mathrm{ev}}_\alpha$, even though its definition depends on the choices we made, and keep in mind that $|\det(\widetilde{\mathrm{ev}}_\alpha)|$ is uniquely determined.

Note that 5.4 gives an example of such a map $\widetilde{\mathrm{ev}}_\alpha$.

Lemma 5.11. *Let C be a curve of degree Δ and relevant and regular type α, which is marked by $\#\Delta + g - 1$ points. Then* $\operatorname{mult}_{\mathrm{ev}(C)}$ *is equal to* $\operatorname{mult} C$ *if C has no string.*

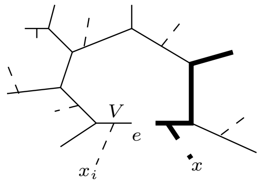
Note that curves with a string are not mapped injectively by ev (see [6], Remark 3.6), therefore they do not contribute to the count $\deg_{\mathrm{ev}}(\mathcal{P})$. Also, if we choose a configuration of points in general position, no curve with a string meets the points.

Proof. We show that $|\det(\widetilde{\mathrm{ev}}_\alpha)|$ equals $\operatorname{mult} C$, which is enough by Remark 5.10. The proof is an induction on the sum of the number of bounded edges and the genus. The induction beginning is shown in [6], Example 3.3.

As induction step, let us now assume C has k bounded edges, is a curve of genus g and degree Δ, and $k + g > 2$. Cut one of the bounded edges. That is, in the graph Γ, choose a bounded edge e and replace it by two ends, each being adjacent to one end vertex of e. Two things can happen:

(1) The graph can decompose into two connected components.
(2) The graph can stay connected, but a loop is broken. We denote the new connected graph of genus $g - 1$ by Γ_1. In this case, the edge e should be chosen such that it is adjacent to a marked point x_i. (Such a choice is possible as C has no string.)

We have to prove the statement for each of the two cases separately, as the arguments differ. The first case is shown in [6], Proposition 3.8. In the second case, Γ_1 has genus $g - 1$, $\#\Delta + 2$ ends that are not marked points, and $\#\Delta + g - 1 < (\#\Delta + 2) + (g - 1) - 1$ marked points, therefore it has a string. This can be seen by removing $\overline{x_i}$ one by one, thus producing several connected components. Since we do not have enough marked points, we end up with less connected components than ends. We add a marked point x adjacent to one of the new ends. There is only one possibility to do this such that the new tropical curve has no string. The tropical curve C_1 of type α_1 defined in this way has genus $g-1$ and as many bounded edges as C. Therefore we can assume by induction that its multiplicity is equal to $|\det \widetilde{\mathrm{ev}}_{\alpha_1}|$. As $\operatorname{mult}(C) = \operatorname{mult}(C_1)$, it remains to show that $|\det \widetilde{\mathrm{ev}}_\alpha| = |\det \widetilde{\mathrm{ev}}_{\alpha_1}|$.



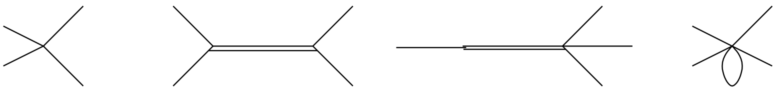
Choose coordinates to compare the two matrices of $\widetilde{\mathrm{ev}}_\alpha$ and $\widetilde{\mathrm{ev}}_{\alpha_1}$. Let V—the vertex adjacent to the marked point x_i—be the root vertex both for C and for C_1. Choose the same order of bounded edges, marked points and loops for the two curves. One of the loops of C, say L, is broken after the cutting of e. This loop corresponds to the last two lines of the matrix of $\widetilde{\mathrm{ev}}_\alpha$. For C_1, the last two lines are given by the marked point x. As chain of flags leading from V to x in C_1, we choose just the same chain of flags as for the loop L. The following table represents both matrices. The two matrices only differ by the $h(V)$-entries in the last two rows. In the table, each row represents two or more rows as before. Each matrix contains the first three rows, $\widetilde{\mathrm{ev}}_\alpha$ contains the fourth, and $\widetilde{\mathrm{ev}}_{\alpha_1}$ the fifth. E_2 denotes the two by two unit matrix.

	$h(V)$	bounded edges
the marked point x_i	E_2	0
other marked points	E_2	$*$
other loops	0	$*$
for $\widetilde{\mathrm{ev}}_\alpha$ the loop L	0	equation for L
for $\widetilde{\mathrm{ev}}_{\alpha_1}$ the new point x	E_2	equation for L

Note that both matrices are block matrices with a 2×2 block on the top left. Therefore, both determinants are equal to the determinant of the lower right block. But this block coincides for both matrices, because it does not involve the two numbers we changed from 0 to 1. $\square$

6. The independence proof

To prove that $\deg_f(\mathcal{P})$ does not depend on $\mathcal{P}$ where f is one of our maps above, note first that the degree is locally constant on the subset of points in f-general position. This is true since at any curve that contributes to $\deg_f(\mathcal{P})$ the map f is a local isomorphism. The points in f-general position are the complement of a polyhedral complex of codimension 1, that is, they form a finite number of top-dimensional regions separated by "walls" that are polyhedra of codimension 1. To show that $\deg_f$ is constant it is therefore enough to consider a general point on such a wall and show that $\deg_f$ is locally constant at these points. Such a general point on a wall is the image under f of a general tropical curve C of a combinatorial type α such that the cell corresponding to α is of codimension 1. We have to classify all those types. For the first enumerative problem, this is done in [11] and for the second in [10]. (For the third, the wall-crossing statement is actually not necessary since all types contribute to the sum $\deg_\delta$, not depending on $\mathcal{P}$.) The following shows local pictures of codimension 1 types.



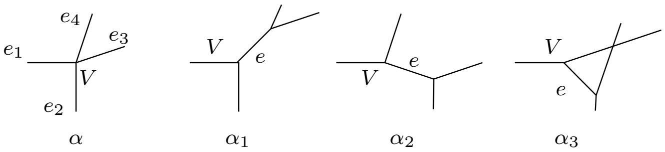
The pictures represent the abstract graph and the direction vectors at the same time: the double edge in the second and third picture from the right represents two edges of the graph Γ which are mapped to the same line segment of $\mathbb{R}^2$ since they are of the same direction. The loop in the picture on the right represents a loop of direction 0 (leading to a type of deficiency 2 which is of codimension 1 because of its 6-valent vertex).

For the first enumerative problem, the numbers $N_{\text{trop}}(d, g)$, we have to consider the left two pictures and the case of a 3-valent curve of genus $g-1$. The other ones are not relevant. For the second enumerative problem, the numbers $E_{\text{trop}}(d, j)$, we have to consider all the pictures above. The third picture from the left leads to several subcases depending on where the marked points are relative to the cycle.

Here, we want to present only the case corresponding to the leftmost picture. This is in fact the most important case, since it has to be considered in all enumerative problems. Also, it is the only case that has to be considered for rational curves. An analogous independence proof for rational curves appeared in [6]. We outline the proof only for the first enumerative problem, the numbers $N_{\text{trop}}(d, g)$, that is, we use the map $\text{ev}\colon \mathcal{M}_{g,n,\text{trop}}(\mathbb{R}^2, \Delta) \to (\mathbb{R}^2)^n$. For the other two maps, it is completely analogous.

Lemma 6.1. *Let $\mathcal{P} \in \mathbb{R}^{2n}$ be a configuration of points such that there is a curve C with one 4-valent vertex satisfying $\text{ev}(C) = \mathcal{P}$. Then the number of preimages near C under ev of a point $\mathcal{P}'$ near $\mathcal{P}$ (counted with multiplicity) does not depend on the choice of $\mathcal{P}'$.*

Proof. Let α be the type of C. The cell $M^{\alpha}_{g,n,\text{trop}}(\mathbb{R}^2, \Delta)$ is in the boundary of three top-dimensional cells, namely the ones where the 4-valent vertex is resolved.



We study the three matrices A_1, A_2 and A_3 of $\widetilde{\text{ev}}_{\alpha_1}$, $\widetilde{\text{ev}}_{\alpha_2}$ and $\widetilde{\text{ev}}_{\alpha_3}$. They differ only in the column corresponding to the edge e. Denote the four edges adjacent to the 4-valent vertex of C with $e_1, \ldots, e_4$, and their respective directions with $v_1, \ldots, v_4$. The root vertex is V as indicated in the picture.

We assume that all choices of flags for evaluation and loops are made consistently. Then the three matrices only differ in the column which belongs to the new edge e. The following table represents all three matrices: Each matrix A_i contains the first block of columns (corresponding to the image $h(V)$ of the root vertex and the lengths l_i of the edges e_i) and the i-th of the last three columns (corresponding to the length of the edge e).

	$h(V)$	l_1	l_2	l_3	l_4	l^{α_1}	l^{α_2}	l^{α_3}
e_1	E_2	v_1	0	0	0	0	0	0
e_2	E_2	0	v_2	0	0	0	$v_2 + v_3$	$v_2 + v_4$
e_3	E_2	0	0	v_3	0	$v_4 + v_3$	$v_2 + v_3$	0
e_4	E_2	0	0	0	v_4	$v_4 + v_3$	0	$v_2 + v_4$
e_1, e_2	0	$-v_1$	v_2	0	0	0	$v_2 + v_3$	$v_2 + v_4$
e_1, e_3	0	$-v_1$	0	v_3	0	$v_3 + v_4$	$v_2 + v_3$	0
e_1, e_4	0	$-v_1$	0	0	v_4	$v_3 + v_4$	0	$v_2 + v_4$
e_2, e_3	0	0	$-v_2$	v_3	0	$v_3 + v_4$	0	$-v_2 - v_4$
e_2, e_4	0	0	$-v_2$	0	v_4	$v_3 + v_4$	$-v_2 - v_3$	0
e_3, e_4	0	0	0	$-v_3$	v_4	0	$-v_2 - v_3$	$v_2 + v_4$

The columns corresponding to the other bounded edges are not shown; it is enough to note here that they are the same for all three matrices. The size of the matrices is $2n + 2g$ times $2 + \#\Gamma_0^1$, and $2 + \#\Gamma_0^1 = 2 + \#\Delta - 3 + 3g = n = \#\Delta + g - 1 + 2g = 2n + 2g$ because of remark 3.4 and definition 5.6. The first four rows correspond to the images of the marked points. The row labeled with e_i stands for the evaluations of marked points that can be reached from V via e_i. The last six rows correspond to the equations of the loops. The row labeled e_i, e_j stands for equations of loops that involve the two edges e_i and e_j. We get four different types of rows for the marked points depending on via which of the four edges e_i a marked point is reached from V. For the loops, we get six different types of rows depending on which two of the four edges $e_1, \dots, e_4$ are involved in a loop. Each row represents in fact two or more rows of the matrix, two rows for the two coordinates of the image of each marked point resp. two equations given by each loop. Loops that do not involve any of the four edges are not added, they do not change the computations. As det is linear in each column, $\det A_1 + \det A_2 + \det A_3$ is equal to the determinant of the following matrix, where we added the three last columns:

	$h(V)$	l_1	l_2	l_3	l_4	
e_1	E_2	v_1	0	0	0	0
e_2	E_2	0	v_2	0	0	$2v_2 + v_3 + v_4$
e_3	E_2	0	0	v_3	0	$2v_3 + v_2 + v_4$
e_4	E_2	0	0	0	v_4	$2v_4 + v_3 + v_2$
e_1 and e_2	0	$-v_1$	v_2	0	0	$2v_2 + v_3 + v_4$
e_1 and e_3	0	$-v_1$	0	v_3	0	$2v_3 + v_2 + v_4$
e_1 and e_4	0	$-v_1$	0	0	v_4	$2v_4 + v_2 + v_3$
e_2 and e_3	0	0	$-v_2$	v_3	0	$v_3 - v_2$
e_2 and e_4	0	0	$-v_2$	0	v_4	$v_4 - v_2$
e_3 and e_4	0	0	0	$-v_3$	v_4	$v_4 - v_3$

Now we subtract the four columns for $l_1, \dots, l_4$ from the last column.

	$h(V)$	l_1	l_2	l_3	l_4	
e_1	E_2	v_1	0	0	0	$-v_1$
e_2	E_2	0	v_2	0	0	$v_2 + v_3 + v_4$
e_3	E_2	0	0	v_3	0	$v_3 + v_2 + v_4$
e_4	E_2	0	0	0	v_4	$v_4 + v_3 + v_2$
e_1 and e_2	0	$-v_1$	v_2	0	0	$v_2 + v_3 + v_4 + v_1$
e_1 and e_3	0	$-v_1$	0	v_3	0	$v_3 + v_2 + v_4 + v_1$
e_1 and e_4	0	$-v_1$	0	0	v_4	$v_4 + v_2 + v_3 + v_1$
e_2 and e_3	0	0	$-v_2$	v_3	0	0
e_2 and e_4	0	0	$-v_2$	0	v_4	0
e_3 and e_4	0	0	0	$-v_3$	v_4	0

Due to the balancing condition $v_1 + v_2 + v_3 + v_4 = 0$. We add v_1 times the $h(V)$-columns to the last column and get a matrix with a zero column whose determinant is 0. Therefore $\det A_1 + \det A_2 + \det A_3 = 0$.

Note that we assume here that the edges e_i are in fact all bounded. If this is not true, the argument needs to be changed slightly. If e_i is unbounded, then there can be no marked points that can be reached from V via e_i. That is, we do not have the corresponding rows.

For a given $i \in \{1, 2, 3\}$ let us now determine whether the combinatorial type α_i occurs in the inverse image under ev of a fixed point $\mathcal{P}'$ near $\mathcal{P}$. We may assume without loss of generality that the multiplicity of α_i is non-zero since other types are irrelevant for the statement of the proposition. Then A_i is an invertible matrix. There is therefore at most one inverse image point. The root vertex and length coordinates for a curve in the inverse image under ev of type α_i are given as $A_i^{-1} \cdot (\mathcal{P}', 0)$, since $\widetilde{\text{ev}}_{\alpha_i} = \text{ev} \times A_{\alpha_i}$ on $M^{\alpha_i}_{g,n,\text{trop}}(\mathbb{R}^2, \Delta)$ by Remark 5.10. In fact, this point exists in $M^{\alpha_i}_{g,n,\text{trop}}(\mathbb{R}^2, \Delta)$ if and only if all coordinates of $A_i^{-1} \cdot (\mathcal{P}', 0)$ corresponding to lengths of bounded edges are positive. By continuity this is obvious for all edges except the newly added edge e, because in the boundary curve C all these edges had positive length. We conclude that there is a curve of type α_i mapping to $\mathcal{P}'$ if and only if the last coordinate (corresponding to the length of the newly added edge e) of $A_i^{-1} \cdot (\mathcal{P}', 0)$ is positive. By Cramer's rule this last coordinate is $\det \widetilde{A}_i / \det A_i$, where $\widetilde{A}_i$ denotes the matrix A_i with the last column replaced by $(\mathcal{P}', 0)$. But note that $\widetilde{A}_i$ does not depend on i since the last column was the only one where the matrices A_i differ. Hence whether there is a curve of type α_i or not depends only on the sign of $\det A_i$: either there are such inverse image points for exactly those i where $\det A_i$ is positive, or exactly for those i where $\det A_i$ is negative. But by the above the sum of the absolute values of the determinants satisfying this condition is the same in both cases. □

Remark 6.2. Note that we have to distinguish a case if we prove an analogous statement for other enumerative problems. If $\Gamma \setminus V$ has two connected components of the same combinatorial type, say the two components containing e_1 and e_2, then the type α_1 gets an extra factor of $\frac{1}{2}$. The types α_2 and α_3 are identical, so the statement is still true in this case. We do not have to consider this case for the numbers $N_{\text{trop}}(d, g)$ or $E_{\text{trop}}(d, j)$, because A_1 would not be injective since we can change length coordinates without changing the image.

7. CONCLUSION

We mentioned in the introduction that we believe that the moduli spaces we consider here should be equipped with tropical structure, and that once this is achieved the tedious case-by-case analysis (for each codimension 1 case) in the independence proof from Section 6 can be replaced by an easy intersection theory argument. This hope is in fact true in the case of rational curves: for rational curves (that is, for the numbers $N_{\text{trop}}(d, 0)$ for example) the moduli space is known to be a fan satisfying a balancing condition as in Remark 2.1 ([5]) and thus a tropical variety. The tropical structure is derived from the tropical structure of $M_{0,n,\text{trop}}$ mentioned in Remark 3.5. A trick has been used to avoid the notion of a tropical stack here: the unmarked (that is, non-contracted) ends are labeled to make them distinguishable even if they have the same direction. Then there is a subgroup G of the symmetric group that acts on the moduli space with labeled ends by relabeling the non-contracted ends. The enumerative numbers we get have to be divided by $|G|$ to reflect the fact that we count each curve several times with different labels for the non-contracted ends ([5]). For a general type, there are $|G|$ ways to relabel the ends. For a type with vertices V such that $\Gamma \setminus V$ has two components of the same type, there are only $\frac{1}{2} \cdot |G|$ ways to label the ends. This enlightens why we include factors of $\frac{1}{2}$ for the weights in such a case in Definitions 4.6 and 4.8.

It is also known that the evaluation map $\text{ev}\colon \mathcal{M}_{0,n,\text{trop}}(\mathbb{R}^2, \Delta) \to (\mathbb{R}^2)^n$ is a morphism of tropical varieties ([5]). Since the space and the map are equipped with tropical structure we can use intersection theory arguments to deduce that $\deg_{\text{ev}}$ is constant for the case of rational curves ([5]).

The proof that the moduli space of rational tropical curves is a tropical variety, that is, balanced ([14], [5]), involves an argument which is very similar to the proof of Lemma 6.1 above. We need to consider a codimension 1 cell (that is, a cell corresponding to a curve with one 4-valent vertex) and consider the neighboring top-dimensional cells just as above. That means that for rational curves, the work to show an independence statement as above is hidden in the proof that the moduli space is a tropical variety. We hope that a similar statement can be shown for higher genus curves, too. We hope that the description of moduli spaces of tropical curves of higher genus as weighted polyhedral complexes used in this survey can contribute to the understanding of their tropical structure.

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