



Tropicalization  
of linear  
isomorphisms on  
plane elliptic  
curves

Arne Buchholz

2 ways to define  
tropical varieties

Duality of plane  
curves

The  $j$ -invariant  
and its tropical  
pendant

Image of a  
tropical  
morphism

$\Delta$ -equivalence

Main statements

# Tropicalization of linear isomorphisms on plane elliptic curves

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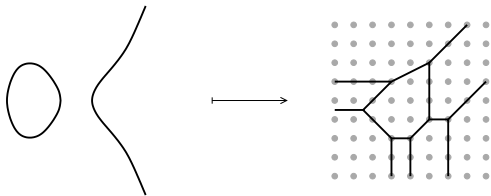
CRCG Higher Order Structures in Mathematics  
Mathematisches Institut der  
Georg-August-Universität Göttingen

Frankfurt, July 16, 2010

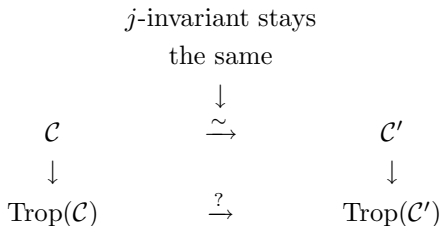


# What we already know

- From classical Algebra:
  - $j$ -invariant determines an elliptic curves up to isomorphy.
  - Elliptic curves can be embedded in  $\mathbb{P}^2$  as cubics.
- E.Katz/H.Markwig/T.Markwig(2008):  
Tropical pendant of  $j$ -invariant is the “cycle length” of a plane tropical curve.



# What we would like to examine



- Questions:
  - What happens to the cycle length?
  - Can we understand the effect directly on the tropical side?
  
- By-product: (generically) tropical linear morphisms **do not** change the  $\Delta$ -class of a curve.

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# What we would like to examine

$$\begin{array}{ccc}
 & j\text{-invariant stays} & \\
 & \text{the same} & \\
 & \downarrow & \\
 \mathcal{C} & \xrightarrow{\sim} & \mathcal{C}' \\
 \downarrow & & \downarrow \\
 \text{Trop}(\mathcal{C}) & \xrightarrow{?} & \text{Trop}(\mathcal{C}')
 \end{array}$$

- Questions:
  - What happens to the cycle length?
    - **“Generically” the cycle length is maintained.**
  - Can we understand the effect directly on the tropical side?

- By-product: (generically) tropical linear morphisms **do not** change the  $\Delta$ -class of a curve.

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- Questions:
  - What happens to the cycle length?
    - **“Generically” the cycle length is maintained.**
  - Can we understand the effect directly on the tropical side?
    - **Yes, we have to consider the corresponding tropical mappings.**
- By-product: (generically) tropical linear morphisms **do not** change the  $\Delta$ -class of a curve.

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# Notations/Conventions

- Classical algebra side:

$\mathbb{K} = \mathbb{C}\{\{t\}\}$ , the field of (formal) **Puiseux series**.

- Elements:

$$p = c_1 t^{q_1} + c_2 t^{q_2} + c_3 t^{q_3} + \dots,$$

where  $c_i \in \mathbb{C}$  and  $q_1 < q_2 < \dots$  live in  $\mathbb{Q}$ .

- **Valuation:**

$$\text{val} : \mathbb{K}^* \rightarrow \mathbb{Q}, (c_1 t^{q_1} + c_2 t^{q_2} + \dots) \mapsto q_1$$

extends to

$$\text{val} : (\mathbb{K}^*)^n \rightarrow \mathbb{Q}^n.$$

(By convention  $\text{val}(0) = \infty$ )

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# Notations/Conventions

- Tropical side: **tropical semi-ring**

$$(\mathbb{T}, \oplus, \odot) := (\mathbb{R} \cup \{-\infty\}, \max, +).$$

- Tropicalization **trop** := **-val** extends naturally to  $\mathbb{K}[\underline{x}]$  via

$$\begin{aligned} \sum_{\alpha} p_{\alpha} x^{\alpha} &\mapsto \bigoplus_{\alpha} \text{trop}(p_{\alpha}) \odot x^{\odot \alpha} \\ &= \max_{\alpha} \{ \langle x, \alpha \rangle - \text{val}(p_{\alpha}) \}. \end{aligned}$$

- 2-dim. **tropical projective torus**:

$$\begin{aligned} \mathbb{R}^3 / \mathbb{R} \cdot (1, 1, 1) &\xrightarrow{1:1} \mathbb{R}^2 \\ (x, y, z) &\mapsto (x - z, y - z). \end{aligned}$$

- “Computations” in  $\mathbb{R}^3$  (e.g. applying mappings).
- Sketch curves in  $\mathbb{R}^2$ .

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# Shades of classical varieties

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Given:

- Ideal  $I \subseteq \mathbb{K}[\underline{x}] := \mathbb{K}[x_1, \dots, x_n]$ .
- Associated (classical) variety  $V = V(I) \subseteq \mathbb{K}^n$ .

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## Definition

The **tropical variety** associated to  $V(I)$  is given by

$$\mathbf{Trop}(V) := \overline{\text{trop}(V \cap (\mathbb{K}^*)^n)}.$$





# Definition of tropical hypersurfaces via tropical polynomials

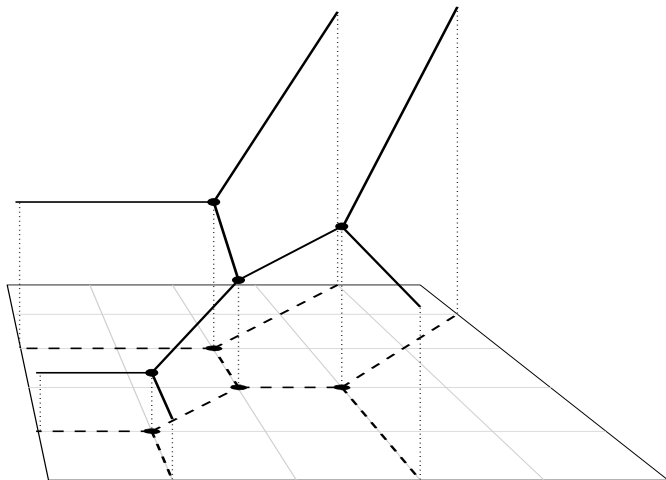
- Situation:  $V = V(I)$  given by **principal ideal**  $I = (f) \neq (0)$ , i.e.  $V$  is a hypersurface.
- For  $f = \sum_{\alpha} p_{\alpha} x^{\alpha}$ : **tropical polynomial** (associated to  $f$ ) is
$$F = \text{trop}(f) = \max_{\alpha} \{ \langle x, \alpha \rangle - \text{val}(p_{\alpha}) \}.$$
- $F$  is a **piece-wise linear**, **convex** function from  $\mathbb{R}^n$  to  $\mathbb{R}$ .

## Definition

For a tropical polynomial  $F$  the **tropical hypersurface**  $\text{Trop}(F)$  is defined to be the **locus of non-linearity** of  $F$  in  $\mathbb{R}^n$ .



# Definition of tropical hypersurfaces via tropical polynomials



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# Subdivision dual to plane curve

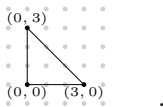
- Given: cubic tropical polynomial

$$F = \max_{0 \leq i+j \leq 3} \{ix + jy + a_{ij}\},$$

where  $a_{i,j} \in \mathbb{T}$ ,  $a_{00}, a_{30}, a_{03} \in \mathbb{R}$ .

- Newton polytope of  $F$ :

$$\mathcal{Q}_3 := \text{conv}\{(i, j) \mid a_{ij} \in \mathbb{R}\}, \text{ i.e.}$$



- $\mathcal{H} := \text{conv}\{(i, j, a_{ij}) \mid a_{ij} \in \mathbb{R}\}$ . The **upper faces** of  $\mathcal{H}$  project surjectively to  $\mathcal{Q}_3$ .  
 $\rightsquigarrow$  **regular (marked) subdivision** of  $\mathcal{Q}_3$ , denoted by  $\Delta_F$ .

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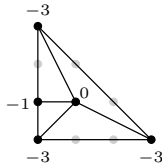
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Example:

$$F = \underbrace{(-3)}_{a_{30}} \odot x^{\odot 3} \oplus \underbrace{(-3)}_{a_{03}} \odot y^{\odot 3} \oplus \underbrace{0}_{a_{11}} \odot x \odot y$$

$$\oplus \underbrace{(-1)}_{a_{01}} \odot y \oplus \underbrace{(-3)}_{a_{00}} :$$



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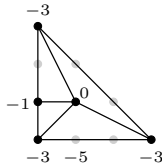
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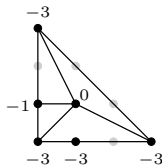
$$F = \underbrace{(-3)}_{a_{30}} \odot x^{\odot 3} \oplus \underbrace{(-3)}_{a_{03}} \odot y^{\odot 3} \oplus \underbrace{0}_{a_{11}} \odot x \odot y \oplus \underbrace{(-5)}_{a_{10}} \odot x \oplus \underbrace{(-1)}_{a_{01}} \odot y \oplus \underbrace{(-3)}_{a_{00}} :$$



# Subdivision dual to a plane curve

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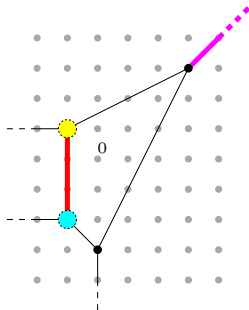
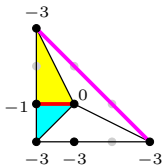
$$F = \underbrace{(-3)}_{a_{30}} \odot x^{\odot 3} \oplus \underbrace{(-3)}_{a_{03}} \odot y^{\odot 3} \oplus \underbrace{0}_{a_{11}} \odot x \odot y \oplus \underbrace{(-3)}_{a_{10}} \odot x \oplus \underbrace{(-1)}_{a_{01}} \odot y \oplus \underbrace{(-3)}_{a_{00}} :$$



Here  $(1, 0)$  is a **marked point** of the subdivision.

# Duality

Subdivision		Tropical Curve $\text{Trop}(F)$
Full-dim. polyhedron	$\leftrightarrow$	Vertex
Interior edge	$\leftrightarrow$	Bounded Edge $\perp$ to $E$
Edges on boundary	$\leftrightarrow$	Rays $\perp$ to $E$
$Q$ edge of $P$	$\leftrightarrow$	$V_P$ lies in $E_Q$



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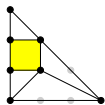
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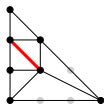
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## Secondary fan

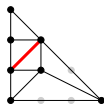
- $\mathcal{A}_3 := \{(i, j) \in \mathbb{N}_0^2 \mid 0 \leq i + j \leq 3\}$  the integer points in  $\mathcal{Q}_3$ .
- Identify:
 
$$F = \max_{i+j \in \mathcal{A}_3} \{ix + jy + a_{ij}\} \leftrightarrow a := (a_{ij}) \in \mathbb{R}^{\mathcal{A}_3}.$$
- Pick subdivision  $\mathcal{S}$ , then  $\{F \mid \Delta_F = \mathcal{S}\}$  forms **polyhedral cone** in  $\mathbb{R}^{\mathcal{A}_3} \cong \mathbb{R}^{10}$ .
- **Secondary fan**: Collection of all these cones.
  - $P$  a face of  $Q \Leftrightarrow \Delta_Q$  “refines”  $\Delta_P$ .
  - **triangulations**: fully refined subdivisions (i.e. belong to **full-dimensional cones** of secondary fan).



is refined by



or



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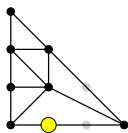
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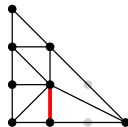
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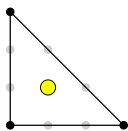
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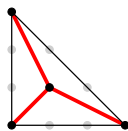
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## $j$ -invariant of an elliptic curve

- **Plane elliptic curve:**

**smooth** curve  $\mathcal{C}_f$  of **genus 1** given by cubic polynomial

$$f = \sum_{i+j \leq 3} p_{ij} x^i y^j \in \mathbb{K}[x, y].$$

- **$j$ -invariant** of  $f$ :

$$j(f) = \frac{A(f)}{\Delta(f)},$$

where  $A, \Delta \in \mathbb{Q}[p]$ ,  $\Delta$  the discriminant.

### Theorem

Two elliptic curves  $\mathcal{C}_f$  and  $\mathcal{C}_g$ , given by cubic polynomials  $f$  and  $g$  respectively, are **isomorphic** as varieties if and only if

$$j(f) = j(g).$$

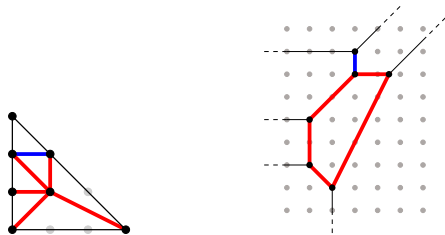
# The cycle length

- **tropical plane elliptic curve**: tropicalization of classical elliptic curve  $\mathcal{C}_f$ , i.e. given by cubic tropical polynomial

$$F = \text{trop}(f) = \max_{i+j \leq 3} \{ix + jy + a_{ij}\}.$$

- $\text{Trop}(F)$  has a **cycle**:  $(1, 1)$  is a **vertex** in the subdivision of  $F$ .
- **Cycle length**: sum of all lattice lengths of cycle segments.

Example:



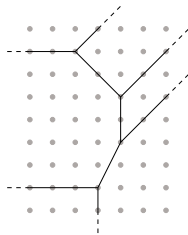
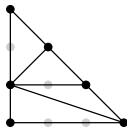
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- $\text{Trop}(F)$  has a **cycle**:  $(1, 1)$  is a **vertex** in the subdivision of  $F$ .
- **Cycle length**: sum of all lattice lengths of cycle segments.

Theorem (E.Katz,H.Markwig,T.Markwig)

If the subdivision  $\Delta_f$  is a *triangulation* of  $\mathcal{Q}_3$ , then

$$\text{cycle length of } \text{Trop}(f) = -\text{val}(j(f)) .$$

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# Image of a tropical morphisms

- Classical morphism of varieties (i.e. polynomial map)

$$\phi = (\phi_1, \dots, \phi_m) : V(I) \rightarrow V(J),$$

all  $\phi_i \in \mathbb{K}[x_1, \dots, x_n]$ .

- **Tropical morphism:**

$$\text{trop}(\phi) := (\text{trop}(\phi_1), \dots, \text{trop}(\phi_m)) : \mathbb{R}^n \rightarrow \mathbb{R}^m.$$

Of course  $\phi(V(I)) \subseteq V(J)$ . It would be nice if we could show

## Theorem

*For the above setting we have*

$$\text{trop}(\phi) (\text{Trop}(I) \setminus \bigcup \text{Trop} \phi_i) \subseteq \text{Trop}(J).$$

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# Image of a tropical linear morphisms on plane elliptic curves

- Situation:  $V(f)$  elliptic curve,  $\phi = \phi_A$  **linear morphism** given by matrix  $A \in \text{GL}_3(\mathbb{K})$ .
- $\phi_A^{-1}(V(f)) = V(f \circ \phi_A)$  has the **same  $j$ -invariant** as  $V(f)$ .
- On tropical side:

$$\text{trop}(\phi_A^{-1})(\text{Trop}(f) \setminus \dots) \subseteq \text{Trop}(f \circ \phi_A).$$

Question: Do  $\text{Trop}(f)$  and  $\text{Trop}(f \circ \phi_A)$  have same cycle length?

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# $\Delta$ -equivalence

- $\Delta$  **discriminant** for cubic polynomials (as before).
- **Gröbner fan**  $\text{Gf}(\Delta)$  of the discriminant: Normal fan of Newton polytope  $\mathcal{Q}_\Delta$  of the discriminant.
- $\text{Gf}(\Delta)$  is a **coarsening** of secondary fan, i.e.  
**each top-dim. cone of the secondary fan lives in exactly one top-dim. cone of  $\text{Gf}(\Delta)$ .**

## Definition

Two subdivisions  $\Delta_1$  and  $\Delta_2$  are  **$\Delta$ -equivalent** if the corresponding cones of the secondary fan lie in the **same top-dim. cone** of the Gröbner fan of  $\Delta$ .

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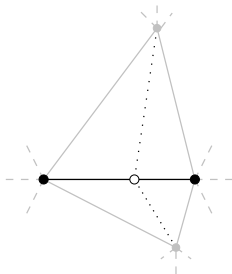
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# $\Delta$ -equivalence geometrically

- Need only 3 types of **modifications** to switch between two triangulations.
  - (1) adding/deleting a “**spoke**”:



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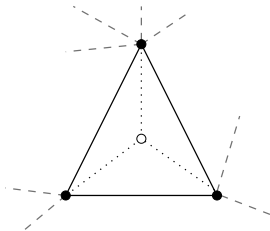
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# $\Delta$ -equivalence geometrically

- Need only 3 types of **modifications** to switch between two triangulations.
  - (2) adding/deleting an “axis”:



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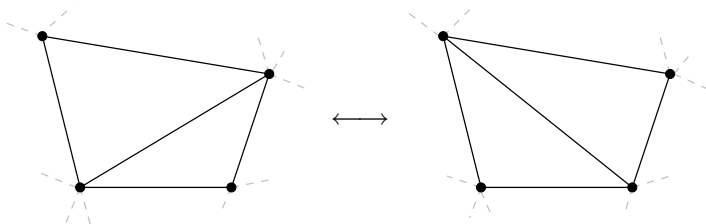
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# $\Delta$ -equivalence geometrically

- Need only 3 types of **modifications** to switch between two triangulations.
- (3) performing a “**flip**”:



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The  $j$ -invariant  
and its tropical  
pendant

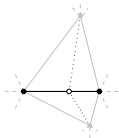
Image of a  
tropical  
morphism

$\Delta$ -equivalence

Main statements

# $\Delta$ -equivalence geometrically

- Need only 3 types of **modifications** to switch between two triangulations.
- (1)



## Corollary (Gelfand, Kapranov, Zelevinsky)

Two triangulation  $\Delta_1$  and  $\Delta_2$  of  $\mathcal{Q}_3$  are  $\Delta$ -equivalent if and only if we can switch between them using **only modification of kind (1)**, where the three points lie on an edge of  $\mathcal{Q}_3$ .

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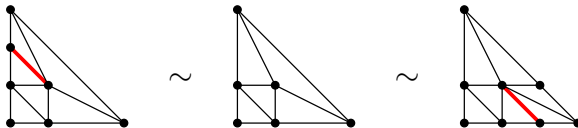
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Example:



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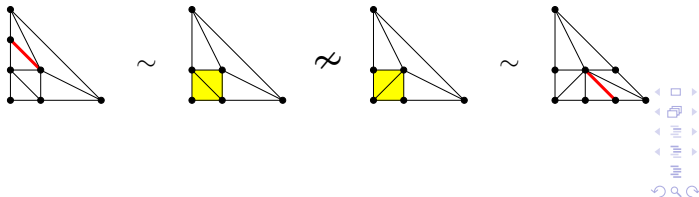
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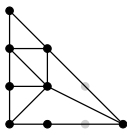
Image of a tropical morphism

$\Delta$ -equivalence

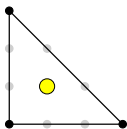
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## 2 further definitions

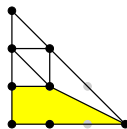
- $A \in \text{Mat}_{n \times n}(\mathbb{K})$  **normal**:  $A = \begin{pmatrix} 1 & & * \\ & \ddots & \\ * & & 1 \end{pmatrix}$  and  $\text{trop}(a_{ij}) \leq 0$  for all  $i, j$ .
- Marked subdivision  $\Delta_f$  is a **weak triangulation**: the maximal cells of  $\Delta_f$  are **triangles** and all marked point lie in 1-cells. e.g.



weak triang.



not a weak tr.



not a weak tr.



# $\Delta$ -equivalence of mapped curves

## Theorem

Assume that  $A \in \mathrm{GL}_3(\mathbb{K})$  is *normal* s.t the *columns of  $\mathrm{trop}(A)$*  do *not* represent the *same point* in tropical projective space and  $f$  is a cubic polynomial over  $\mathbb{K}$ , defining an elliptic curve, s.t.

- $\mathrm{Trop}(f)$  and  $\mathrm{Trop}(f \circ \phi_A)$  have a *cycle*,
- $\Delta_f$  and  $\Delta_{f \circ \phi_A}$  are *weak triangulations* and
- $\mathrm{trop}(f \circ \phi_A) = \mathrm{trop}(f) \circ \mathrm{trop}(\phi_A)$ , i.e. we can compose  $\phi_A$  with  $f$  *without cancellation* of lowest order terms.

*Then  $\Delta_f$  and  $\Delta_{f \circ \phi_A}$  are  $\Delta$ -equivalent.*



# Sketch of the proof

- Decompose  $A$  into easy matrices, s.t. decomposition is compatible with tropicalization.
- Examine effect of such matrices on subdivision.
- Show: Such matrices can only provide the desired modifications.

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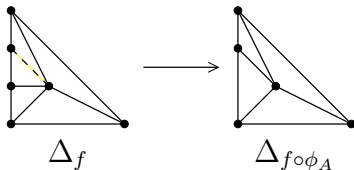


# Remarks to the theorem

Under the assumptions of the above theorem:

## Remark

An edge in  $\Delta_f$  can *only be deleted* by  $\phi_A$  if its “neighbour” is visible after mapping.



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Under the assumptions of the above theorem:

### Remark

An edge in  $\Delta_f$  can *only be deleted* by  $\phi_A$  if its “neighbour” is visible after mapping.

### Remark

If  $A$  is not normal the subdivisions are  $\Delta$ -equivalent “up to permutation”.

### Remark

If an edge appears in  $\Delta_{f \circ \phi_A}$ , i.e. a point on the boundary of  $\mathcal{Q}_3$  is lifted, then the term which yields the *lifting* “comes from” the term to  $(1, 1)$  in  $f$ .

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# Linear morphisms maintain cycle length

## Theorem

Assume that  $A \in \mathrm{GL}_3(\mathbb{K})$ , s.t. the columns of  $\mathrm{trop}(A)$  do *not* represent the *same point* in the tropical projective plane. Let  $f$  define an elliptic curve, s.t.

- $\mathrm{Trop}(f)$  has a *cycle*,
- $\Delta_f$  and  $\Delta_{f \circ \phi_A}$  are *weak triangulations*,
- the closure of  $\mathrm{trop}(\phi_A^{-1}) (\mathrm{Trop}(f) \setminus \bigcup (A^{-1})_i)$  has a *“cycle”*.

Then  $\mathrm{Trop}(f)$  and  $\mathrm{Trop}(f \circ \phi_A)$  have the *same cycle length*.

## Remark

Cycle length does not have to have expected value in the theorem.

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# Outlook

## Questions/Tasks:

- Weaken as much preconditions as possible. (e.g.: Does the image of  $\text{trop}(\phi_A^{-1})$  really need to have a cycle?)
- Can we generalize the results to understand  $\Delta$ -equivalence in higher dimensions?
- We (probably) can use “tropical modification” to make missing cycle length visible (if a curve does not have the expected length). How do linear morphisms effect this part of the cycle?

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Thank you for your  
attention!

