Diffeology (or another point of view to differential geometry)

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I Diffeological spaces

- Philosophy: Diffeological spaces are what people (probably) had in mind when doing differential geometry before the concept of a manifold was invented.
- Examples: $C^{\infty}(\mathbb{R},\mathbb{R})$ and $T_{\alpha} := \mathbb{R}/(\mathbb{Z}+\alpha\mathbb{Z})$ are (for $\alpha \notin \mathbb{Q}$) no manifolds, but very natural and good objects to consider!
- Guideline: a smooth family in $C^{\infty}(\mathbb{R},\mathbb{R})$ should be an assignment $(a,b) \ni s \mapsto f_s \in C^{\infty}(\mathbb{R},\mathbb{R})$ such that $(a,b)\times\mathbb{R}\to\mathbb{R}$, $(s,x)\mapsto f_s(x)$ are smooth.

For X a set

$$\mathrm{Param}(X) = \cup_{n=0}^{\infty} \cup_{U\in\mathrm{Top}(\mathbb{R}^n)} \mathrm{Map}(U,X)$$

The a subset $\mathcal{D}\subseteq \mathrm{Param}(X)$ is supposed to encode "smooth parametrisations" of X if

1. (Cover axiom): X is covered by the values of smooth parametrizations, i.e.

$$\sup_{P\subseteq\mathcal{D}} P(\mathrm{dom(P)}) = X$$

2. (smooth compatibility): for $P\in\mathcal{D}$ and for all $F\in C^{\infty}(V,U)$ we have $P\circ F\in\mathcal{D}$
3. (locality): $P: U\to X$, if for each $r_0\in\mathrm{dom}(P)$ there exists an open superset of r_0 such that $P|_V\in\mathcal{D}$, then $P\in\mathcal{D}$.

Example I.1.

$$\begin{array}{ccc} & & US^2 := \{(x,v)\in S^2\times S^2 \mid x\cdot v=0\} \\ & \overset{Q}{\nearrow} & \downarrow \\ \mathbb{R}^2-0 & \xrightarrow[p]{} & S^2 \end{array}$$

... ■

Definition I.2. A diffeology $\mathcal{D}$ on a set X is a subset $\mathcal{D}\subseteq\mathrm{Param}(X)$ satisfying the above axioms. The we sall $\mathcal{D}$ also a smooth structure on X and $(X,\mathcal{D})$ a *diffeological space*. A morphism is a map $f: X\to X'$ such that $f\circ\mathcal{D}\subseteq\mathcal{D}'$. ■

Example I.3.
- num domain
- manifolds
- T_{α} (as introduced above) ■

Isomorphisms (or diffeomorphisms) are bijective maps $f: X\to X'$ such that f^{-1} is also smooth.

- if $\alpha\in\mathbb{Q}$, then $T_{\alpha}\simeq\mathbb{R}/\mathbb{Z}$

- In the case of T_α, we have $T_\alpha \cong T_\beta$ iff $\alpha \sim_{\mathrm{GL}_2(\mathbb{Z})} \beta$ (i.e. $\beta = \frac{a\alpha+b}{c\alpha+d}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$).
- $$\mathrm{Diff}(T_\alpha) = \begin{cases} T_\alpha \times \mathbb{Z}_2 \times \mathbb{Z} & \text{if } \alpha \text{ is a quadratic number} \\ T_\alpha \times \mathbb{Z}_2 \times \{0\} & \text{else} \end{cases}$$
- $\forall \alpha \in \mathbb{R} - \mathbb{Q}$, $C^\infty(T_\alpha, \mathbb{R}) = \mathbb{R}$ (as diffeological spaces ???)

Properties:

1. $C^\infty(X, X')$ is naturally a diffeological space and we have
$$C^\infty(X, C^\infty(X', X'')) \cong C^\infty(X \times X', X'')$$
2. have products: $X \times X'$
3. quotients: $X \xrightarrow{\pi} X/\sim$ (then $\mathcal{D}$ induces $\pi_*(\mathcal{D})$)
4. $A \hookrightarrow X$

What we have for diffeological spaces:

- Homotopy theory (connectedness), $\pi_k(X, x)$, $\pi_k(X, x) = \pi_{k-1}(\Omega X, \overline{x})$
- differential calculus, differential forms α, $d\alpha \rightsquigarrow$ moment maps
- fibre bundles, exapt homotopy sequences
- coverings

Competitives approaches:

- Frölicher spaces
- Differneial spaces (Sikorsky)

II Fibrations in diffeology

Definition II.1. A diffeological group is a group G, together with a diffeology such that the multiplication and inversion maps are smooth. A *diffeological groupoid* is a groupoid K such that $\mathrm{Ob}(K)$, $\mathrm{Mor}(K)$ are diffeological spaces such that

a) ends := source × target : $\mathrm{Mor}(K) \to \mathrm{Ob}(K)$ is smooth

b) mulriplication $\mathrm{Mor}(K)_s \times_t \mathrm{Mor}(K) \ni (f, g) \mapsto fg \in \mathrm{Mor}(K)$ is smooth

c) inversion $\mathrm{Mor}(K) \ni f \mapsto f^{-1} \in \mathrm{Mor}(K)$ is smooth ∎

Example II.2. X diffeological, $\mathrm{Diff}(X)$ with the functional diffeology:

$$\begin{cases} P : U \to \mathrm{Diff}(X) \\ (r, x) \mapsto P(r)(x) \in C^\infty(U \times X, X) \\ (r, x) \mapsto P(r^{-1})(x) \in C^\infty(U \times X, X) \end{cases}$$

$\mathrm{Diff}(X) \times X \to X$ $(f, x) \mapsto f(x)$ ∎

Definition II.3. A diffeological groupoid ix *fibrating* iff ends is a subduction (it is surjective and local lifts for plots exists). ∎

Let $X \xrightarrow{\pi} Q$ be a projection, i.e. $Q = X/\sim$. Then we obtain a diffeological groupoid $K(\pi)$ when setting

- $\mathrm{Obj}(K) = Q$
- $\mathrm{Mor}(K) = \{f \in \mathrm{Diff}(q, q') | q, q' \in Q\}$

Endowing $\mathrm{Mor}(K)$ with the coarsest diffeology such that ev and $\overline{\mathrm{ev}}$ are smooth, where ev and $\overline{\mathrm{ev}}$ are defined by

$$\{(f, x) | x \in \mathrm{dom}(f)\} \xrightarrow{\mathrm{ev}} (x, f(x)) \quad \{(f, x) | x \in \mathrm{dom}(f^{-1})\} \xrightarrow{\overline{\mathrm{ev}}} (x, f^{-1}(x))$$

...some definition of plots...

Definition II.4. $\pi : X \to Q$ is a *diffeological fiber bundle* iff $K(\pi)$ is fibrating. ∎

Proposition II.5. *$\pi : X \to Q$ is a fibration iff for each plot $P : U \to X$, the pull-back (in* Set*) of this fibration to a fibration over U is locally trivial.*

Example II.6. $T^2 \to T_\alpha$, (defined in the obvious way) is a fibration with structure group $(\mathbb{R}, +)$. ∎

Definition II.7. A prinicpal fibration is a fibration $T \to X$, together with a diffeological action of some diffeological group G on T such that there exist equivariant local isomorphisms. In the obvious way one defines associated bundles (with the quotient diffeology). ∎

Proposition II.8. *Every fiber bundle is associated to a principal bundle.*

II.1 Homotopy sequences

X: diff. space, $\rightsquigarrow \pi_0(X)$, $\pi_k(X, x) = \pi_{k-1}(\Omega X, \overline{x})$ (higher homotopy groups)

Proposition II.9. *For each fibration $T \to Q$ one has a long exact homotopy sequence.*

Proposition II.10. *Every diffeological fiber bundle over $\mathbb{R}^n$ is trivial.*

Proposition II.11. *If G is a diffeological group and H is any subgroup, then $G \to G/H$ is a diffeological principal bundle.*

(n.b.: $T^2 \to T_\alpha = T^2/\mathbb{R}$ is an example for the latter proposition.)

Coverings: X with $\pi_0(X) = 0$, then a covering of X is a fibrations $\widehat{X} \to X$ such that the fiber F is diffeologically discrete.

Theorem II.12. *For any connected X, there exists an (up to isom. unique) cover $\widehat{X} \to X$ with fiber $\pi_1(X)$ and $\widehat{X}$ simply connected. Any other covering is a quotient of this universal covering.*

Proof. Via the fibrating path groupoid $\mathrm{Paths}(X) \xrightarrow{\hat{0} \times \hat{1}} X \times X$. ∎

There exists $K : \Omega^p(X) \to \Omega^{p-1}(\mathrm{Paths}(X))$ s.th. $K \circ d + d \circ K = \hat{1}^* - \hat{0}^*$. If $X \xrightarrow{f_0, f_1} X'$ and f_s a homotopy between f_0 and f_1, then consider

$$\Phi : X \to \mathrm{Paths}(X'), \quad X \mapsto [s \mapsto f_s(x)].$$

With this, we have

$$\Phi^*(Kd\alpha + dK\alpha) = \Phi^*(\hat{1}^*(\alpha) - \hat{0}^*(\alpha)) = d\beta(\hat{1} \circ \Phi)^*\alpha - (\hat{0} \circ \Phi)^*\alpha = f_1^*\alpha - f_0^*\alpha$$

and one obtains the homotopy invariance of de Rham cohomology.

In the end: ...some construction of prequantum bundles, probably the same as in "La trilogie du moment"...