Twisted Geometric Cycles

These talks are on some results of my collaborator Bai-Ling Wang in arXiv:0710.1625: Geometric cycles, index theory and twisted K-homology. (Journal of NCG 2008)

It had its origins in our joint work:
arXiv:0708.3114: Differential Twisted K-theory and Applications (with Jouko Mickelsson and Bai-Ling Wang)

and in
arXiv:math/0507414: Thom isomorphism and Push-forward map in twisted K-theory (with Bai-Ling Wang)

I will touch on some applications to string theory that are discussed in detail in
Riemann-Roch and index formulae in twisted K-theory, (with B-L Wang), Symposia in Pure Mathematics,**81** (2010) 95-131, American Mathematical Society, Providence RI

Overview

There are several ingredients that need explaining from the topological and geometric side that are quite recent. (The analytic side is much older and I will not say much about it in these talks.)

(i) Twisted K theory.

(ii) Twisted Poincaré duality between twisted K-cohomology and twisted K-homology.

(iii) Generalising Baum-Douglas K-homology to the twisted situation.

I am deliberately going to suppress the use of gerbes in these talks. They are used to give simple proofs of some of the technical results but I will omit those discussions.

Motivation and applications

Our aim was to formulate in a precise mathematical way Witten's ideas on 'D-brane charges' as taking values in twisted K-theory and to get a twisted version of some of what is in his original article hep-th/9810188 'D-branes and K-theory'.

The end result of Wang's work is to justify the view that D-branes in the sense used by Witten are twisted Baum-Douglas geometric cycles. In the absence of a twist Szabo explained how ordinary D-branes may be thought of as Baum-Douglas geometric cycles. The main obstruction to generalising his viewpoint was the absence of an appropriate notion of twisted geometric cycles until the work of Wang.

Other motivating factors are in the work of Brodski, Mathai, Rosenberg, Szabo, Adv. Theor. Math. Phys. **13** (2009) 497-552, 'Noncommutative correspondences, duality and D-branes in bivariant K-theory'. Here the analytic point of view of twisted K-Homology is developed using Kasparov's theory.

There are other applications.

These are to twisted index formulas and to twisted versions of the Atiyah-Hirzebruch Riemann-Roch theorem.

There is a generalisation of these ideas to string geometry that has recently led to an understanding of Witten's quantisation condition in M-theory.

I will survey some of these very briefly (they are discussed in the previous references).

1. Twisted K-theory: topological and analytic definitions

Let X be a closed manifold, and $\mathcal{H}$ be an infinite dimensional, complex and separable Hilbert space.

$PU(\mathcal{H})$ is the projective unitary group with norm topology. $PU(\mathcal{H})$ can be identified with an Eilenberg-MacLane space $K(\mathbb{Z}, 2)$. So the classifying space $BPU(\mathcal{H})$ is a $K(\mathbb{Z}, 3)$.

A twisting is a continuous map $\alpha : X \to K(\mathbb{Z}, 3)$. The associated $PU(\mathcal{H})$ bundle $\mathcal{P}_\alpha$ is given by pulling back the universal $PU(\mathcal{H})$-bundle over $K(\mathbb{Z}, 3)$.

The set of isomorphism classes of principal $PU(\mathcal{H})$-bundles over X is the homotopy classes of maps

$$[X, K(\mathbb{Z}, 3)] \cong H^3(X, \mathbb{Z}).$$

Let $\mathbf{Fred}$ be the space of Fredholm operators with norm topology.

The 'conjugation' action $PU(\mathcal{H}) \times \mathbf{Fred} \longrightarrow \mathbf{Fred}$ defines an associated bundle with fiber the Fredholm operators

$$\mathcal{P}_\alpha(\mathbf{Fred}) = \mathcal{P}_\alpha \times_{PU(\mathcal{H})} \mathbf{Fred}$$

Let $\Omega^n_X \mathcal{P}_\alpha(\mathbf{Fred}) = \mathcal{P}_\alpha \times_{PU(\mathcal{H})} \Omega^n \mathbf{Fred}$ be the fiberwise n-iterated loop spaces.

The (topological) twisted K-groups of (X, α) are defined to be

$$K^{-n}(X, \alpha) := \pi_0\Big(C_c(X, \Omega^n_X \mathcal{P}_\alpha(\mathbf{Fred}))\Big),$$

the set of homotopy classes of compactly supported sections. Due to Bott periodicity, we only have two different twisted K-groups, denoted by $K^0(X, \alpha)$ and $K^1(X, \alpha)$.

2. Twisted K-homology: Analytic and topological definitions

The analytic version can be obtained as follows.

Associated with the $PU(\mathcal{H})$ bundle $\mathcal{P}_\alpha$ is a continuous trace C*-algebra and one may define the analytic twisted K-theory (resp. twisted K-homology) of (X, α) as the K-theory (resp K-homology) of this algebra. This is discussed in BMRS and uses Kasparov's KK theory point of view from operator algebra theory. It goes back to a paper of Rosenberg in 1987.

The analytic twisted K-homology of (X, α), will be denoted by

$$K^{\mathbf{an}}_{ev/odd}(X, \alpha)$$

.

Next introduce the space $\mathcal{P}_\alpha(\mathbf{Fred})/X$ obtained by identifying the base points (the identity operator) in the fibers. Then the topological twisted K-homology $K^{\mathbf{top}}_{ev/odd}(X,\alpha)$ is defined to be

$$K^{\mathbf{top}}_{ev}(X,\alpha) = \varinjlim_{k\to\infty} \pi_{2k}\Big(\mathcal{P}_\alpha(\mathbf{Fred})/X\Big)$$

and

$$K^{\mathbf{top}}_{odd}(X,\alpha) = \varinjlim_{k\to\infty} \pi_{2k+1}\Big(\mathcal{P}_\alpha(\mathbf{Fred})/X\Big).$$

The proof that the topological and analytic objects are isomorphic uses twisted Poincare dualities in the topological and analytic settings and the equivalence between topological and analytic twisted K-theory.

3. Twisted Poincaré duality

The twisted version introduces a shift in the twist

$$\alpha \mapsto \alpha + (W_3 \circ \tau)$$

where $\tau : X \to BSO$ is the classifying map of the stable tangent bundle and W_3 is the classifying map for the bundle $\mathbf{BSpin^c} \to \mathbf{BSO}$, and $\alpha + (W_3 \circ \tau)$ denotes the map $X \to K(\mathbb{Z}, 3)$, representing the class $[\alpha] + W_3(X)$ in $H^3(X, \mathbb{Z})$. (There is a tricky point in this definition for which I refer back to the original paper of Wang.)

Theorem. Let X be a smooth manifold with a twisting $\alpha : X \to K(\mathbb{Z}, 3)$.
(i) (Wang) There exists an isomorphism

$$K^{\mathbf{top}}_{ev/odd}(X, \alpha) \cong K^{ev/odd}_{\mathbf{top}}(X, \alpha + (W_3 \circ \tau))$$

with the degree shifted by $dim X (mod\ 2)$.
(ii) (Tu, Echterhoff-Emerson-Kim) There exists an isomorphism

$$K^{\mathbf{an}}_{ev/odd}(X, \alpha) \cong K^{ev/odd}_{\mathbf{an}}(X, \alpha + (W_3 \circ \tau))$$

with the degree shifted by $dim X (mod\ 2)$.

4. Twisted geometric cycles

Let (X, α) be a closed manifold with a twisting α.

A geometric cycle for (X, α) is a quintuple

$$(M, \iota, \nu, \eta, [E])$$

where $[E]$ is a K-class in $K^0(M)$, M an oriented smooth closed manifold with a classifying map ν of its stable normal bundle, $\iota : M \to X$ is a continuous map such that there exists a homotopy commutative diagram:

$$\begin{tikzcd}
M \arrow[r, "\nu"] \arrow[d, "\iota"'] & \mathbf{BSO} \arrow[d, "W_3"] \arrow[dl, Rightarrow, dashed, "\eta"'] \\
X \arrow[r, "\alpha"] & K(\mathbb{Z}, 3),
\end{tikzcd}$$

with a homotopy η between $W_3 \circ \nu$ and $\alpha \circ \iota$. We refer to this diagram of maps as an 'α-twisted Spin^c structure'.

Remarks. (i) For the map on the right hand side ($\mathbf{BSO} \to K(\mathbb{Z}, 3)$) we just use the definition of W_3.

(ii) Wang and I had noted in our earlier work that M admits an α-twisted $Spin^c$ structure if and only if

$$\iota^*([\alpha]) + W_3(M) = 0.$$

If ι is an embedding, this is the anomaly cancellation condition introduced by Freed and Witten for physical reasons (well-definedness of the world sheet action). Thus we found the mathematical significance of this condition.

(iii) Importantly if the twists are all trivial this reduces to the Baum-Douglas definition and η corresponds to a choice of Spin^c structure.

Question: do these twisted geometric cycles provide those D-branes that are relevant for string theory?

From physical arguments we know that there is some kind equivalence relation on D-branes. Mathematically we try to capture this by using the ideas of Baum-Douglas.

Two geometric cycles $(M_1, \iota_1, \nu_1, \eta_1, [E_1])$ and $(M_2, \iota_{,2} \nu_2, \eta_2, [E_2])$ are isomorphic if there is an isomorphism $f : (M_1, \iota_1, \nu_1, \eta_1) \to (M_2, \iota_2, \nu_2, \eta_2)$, as α-twisted $Spin^c$ manifolds over X, such that $f_!([E_1]) = [E_2]$.

Let $\Gamma(X, \alpha)$ be the collection of all geometric cycles for (X, α). We now impose an equivalence relation $\sim$ on $\Gamma(X, \alpha)$.

It is generated by three relations that are simple adaptations of the original relations of Baum and Douglas (and for a detailed explanation of these it is best to go back to their papers).

Direct sum - disjoint union

If $(M, \iota, \nu, \eta, [E_1])$ and $(M, \iota, \nu, \eta, [E_2])$ are two geometric cycles with the same α-twisted $Spin^c$ structure, then

$$(M, \iota, \nu, \eta, [E_1]) \cup (M, \iota, \nu, \eta, [E_2])$$

$$\sim (M, \iota, \nu, \eta, [E_1] + [E_2]).$$

Bordism

Let $-(M_1, \iota_1, \nu_1, \eta_1)$ denote the manifold M_1 with the opposite α-twisted $Spin^c$ structure (needs a slightly non-trivial construction). If there exists an α-twisted $Spin^c$ manifold (W, ι, ν, η) and $[E] \in K^0(W)$ such that

$$\partial(W, \iota, \nu, \eta) = -(M_1, \iota_1, \nu_1, \eta_1) \cup (M_2, \iota_2, \nu_2, \eta_2)$$

and $\partial([E]) = [E_1] \cup [E_2]$ then

$$(M_1, \iota_1, \nu_1, \eta_1, [E_1]) \sim (M_2, \iota_2, \nu_2, \eta_2, [E_2]).$$

$Spin^c$ vector bundle modification

Take a geometric cycle $(M, \iota, \nu, \eta, [E])$ and a $Spin^c$ vector bundle V over M with even dimensional fibers. Denote by $\underline{\mathbb{R}}$ the trivial rank one real vector bundle. Choose a Riemannian metric on $V \oplus \underline{\mathbb{R}}$, let

$$\widehat{M} = S(V \oplus \underline{\mathbb{R}})$$

be the sphere bundle of $V \oplus \underline{\mathbb{R}}$.

Denote by $\rho : \widehat{M} \to M$ the projection which is K-oriented. The vertical tangent bundle $T^v(\widehat{M})$ of $\widehat{M}$ admits a natural $Spin^c$ structure with an associated $\mathbb{Z}_2$-graded spinor bundle $S_V^+ \oplus S_V^-$. Then

$$(M, \iota, \nu, \eta, [E]) \sim (\widehat{M}, \iota \circ \rho, \nu \circ \rho, \eta \circ \rho, [\rho^* E \otimes S_V^+]).$$

This notion builds in Bott periodicity.

Definition. The geometric twisted K-homology $K^{\mathbf{geo}}_{ev/odd}(X,\alpha)$ is defined to be $\Gamma(X,\alpha)/\sim$ with the grading given by even or odd dimension of α-twisted $Spin^c$ manifolds.

Addition is given by the disjoint union - direct sum relation.

From the point of view of physics it is vector bundle modification that looks rather hard to explain.

4. Twisted assembly map

There exists a natural homomorphism $\mu : K^{\mathbf{geo}}_{ev/odd}(X,\alpha) \to K^{\mathbf{an}}_{ev/odd}(X,\alpha)$ where $\mu(M,\iota,\nu,\eta,[E])$ is defined by composition of a sequence of maps:

$$\begin{tikzcd}
& {[E] \in K^0(M)} \arrow[r, "PD"] & K^{\mathbf{an}}_{ev/odd}(M, W_3\circ\tau) \arrow[d, "I_*"] \\
& K^{\mathbf{an}}_{ev/odd}(M,\alpha\circ\iota) \arrow[d, "\iota_*", "\cong"'] & K^{\mathbf{an}}_{ev/odd}(M, W_3\circ\nu) \arrow[l, "\cong", "\eta_*"'] \\
& K^{\mathbf{an}}_{ev/odd}(X,\alpha). &
\end{tikzcd}$$

Here $PD : K^0(M) \cong K^{\mathbf{an}}_{ev/odd}(M, W_3\circ\tau)$ is Kasparov's Poincaré duality with the degree shift by $dim M(mod\ 2)$, ι_* is the natural push-forward map in twisted K-homology, η_* is the isomorphism induced by the homotopy η, and I_* is the isomorphism induced by the trivial $Spin^c$ structure on the trivial bundle $\tau\oplus\nu$.

Theorem (Wang) The twisted assembly map

$$\mu : K^{\mathbf{geo}}_{ev/odd}(X,\alpha) \to K^{\mathbf{an}}_{ev/odd}(X,\alpha)$$

is an isomorphism for any **smooth** closed manifold X with a twisting $\alpha : X \to K(\mathbb{Z},3)$.

The proof of this theorem is via establishing that there is a map $\Psi : K^{\mathbf{top}}_{ev}(X,\alpha) \to K^{\mathbf{geo}}_{0}(X,\alpha)$ such that the following diagram

$$\begin{tikzcd}
 & K^{\mathbf{top}}_{ev/odd}(X,\alpha) \arrow[dl, "\Psi"'] \arrow[dr, "\Phi", "\cong"'] & \\
K^{\mathbf{geo}}_{ev/odd}(X,\alpha) \arrow[rr, "\mu"] & & K^{\mathbf{an}}_{ev/odd}(X,\alpha)
\end{tikzcd}$$

commutes and Ψ is surjective.

5. The twisted index theorem.

One of the applications of geometric cycles is to express an index pairing between twisted K-theory and twisted K-homology in terms of an index pairing on geometric cycles.

This leads eventually to a twisted counterpart to a formula first found by Minasian and Moore in the untwisted case.

Theorem. Let X be a smooth closed manifold with a twisting $\alpha : X \to K(\mathbb{Z}, 3)$. The index pairing

$$K_0(X, \alpha) \times K^0(X, \alpha) \longrightarrow \mathbb{Z}$$

is given by

$$< (M, \iota, \nu, \eta, [E]), \xi >= \int_M ch_{w_2(M)}\Big(\eta_*(\iota^* \xi \otimes E)\Big)\widehat{A}(M)$$

where $\xi \in K^0(X, \alpha)$, and the geometric cycle

$$(M, \iota, \nu, \eta, [E])$$

defines a twisted K-homology class on (X, α).

To explain this formula we have to unpack it further.

Recall the definition of ‘α-twisted Spin^c structure’:

$$\begin{tikzcd}
M \arrow[r, "\nu"] \arrow[d, "\iota"'] & \mathbf{BSO} \arrow[d, "W_3"] \arrow[dl, Rightarrow, dashed, "\eta"'] \\
X \arrow[r, "\alpha"] & K(\mathbb{Z},3),
\end{tikzcd}$$

We use this diagram to understand $\iota^*\xi$ which is an element of twisted K-theory of M.

We know what the twist is because

$$\eta_* : K^*(M, \iota^*\alpha) \cong K^*(M, W_3(M))$$

is an isomorphism.

Next we remark that $K^0(M)$ acts on twisted K-theory and abstractly we can think of this as being represented by $\iota^*\xi \otimes E$.

However, there is a concrete model for $K^*(M, W_3(M))$ which we also use to understand $ch_{w_2(M)}$ which is the twisted Chern character on $K^0(M, W_3(M))$.

6. Twisted Chern character

There is a well known identification between $K^0(M, W_3(M))$ and the K-theory of Clifford bundles over M. (A Clifford bundle is one whose fibre is a Clifford algebra but whose structure group is a subgroup of a projective unitary group.)

In fact $W_3(M)$ can be constructed as the obstruction to the existence of an isomorphism of Cliifford bundle over M to a vector bundle. (It is the simplest case of a lifting bundle gerbe where we want to lift the structure group for the Clifford bundle to the group $U(n)$ for some n.)

Now we think of $\iota^*\xi \otimes E$ as the tensor product of vector bundles but $\iota^*\xi$ is represented by a Clifford bundle with an action of the Clifford algebra so that the fibre of $\iota^*\xi \otimes E$ is a Clifford module.

The index theorem for such Clifford modules is described in the book of Berline-Getzler-Vergne.

In this context the map

$$ch_{w_2(M)} : K^0(M, W_3(M)) \longrightarrow H^{ev}(M, \mathbb{R})$$

is given by the relative Chern character on Clifford bundles written in the notation $ch(F/S)$ by Berline-Getzler-Vergne where S is the non-existent spin bundle on M and F/S is the curvature of a Clifford connection on $\iota^*\xi \otimes E$ regarded as a Clifford module. The point is that $\widehat{A}(M)$ need not represent an integral class but that $ch(F/S)$ is defined so as to correct this defect and give an integer when inserted into the integral:

$$\int_M ch_{w_2(M)}\Big(\eta_*(\iota^*\xi \otimes E)\Big)\widehat{A}(M)$$

Thus the strength of Wang's theorem is to give, via Berline-Getztler Vergne, an elliptic operator on a Clifford bundle over M whose class in $K^0(M, W_3(M))$ pushes forward to a class in $K^0(X, \alpha)$ and by considering all geometric cycles we exhaust $K^0(X, \alpha)$.

There is no such simple definition of the twisted Chern character for other twists.

The general twisted Chern character on $K^0(X, \alpha)$ requires a choice of gerbe connection and curving. A geometric definition was given in *Differential Twisted K-theory and its Applications*, C-Mickelsson-Wang.

An analytical definition using the Chern-Connes character in noncommutative geometry was given Mathai-Stevenson. A topological definition was given by Atiyah-Segal.

7. Twisted Riemann-Roch

The previous formula is not the analogue of Minasian-Moore. We need to rewite it on X.

To do this we have to use a Riemann-Roch theorem in C-Mickelsson-Wang *op cit*, which implies that the above index formula can be written as

$$\begin{aligned} & < (M, \iota, \nu, \eta, [E]), \xi > \\ = & \int_M ch_{w_2(M)}\Big(\eta_*(\iota^*\xi \otimes E)\Big)\widehat{A}(M) \\ = & \int_X ch_{w_2(X)}\Big(\ \iota_!(E) \otimes \xi\Big)\widehat{A}(X). \end{aligned}$$

I will partially unpack this formula.

The tricky bits are $\iota_!$ and $ch_{w_2(X)}$. The first is the push-forward map on twisted K-theory (in general defined by C-Wang much earlier).

In this context we think of it as follows

$$\begin{aligned} K^0(M) &\cong K_0(M, W_3(M)) \\ &\cong K_0(M, -\iota^*\alpha) \\ &\mapsto K_0(X, -\alpha) \\ &\cong K^0(X, -\alpha + W_3(X)). \end{aligned}$$

Now $ch_{w_2(X)}$is complicated and this is the key point of our theorem. It may be defined as a ‘twisted Chern character map’ sending

$$K^0(X, -\alpha + W_3(X)) \otimes K^0(X, \alpha) \to K^0(X, W_3(X)).$$

Notice that the group on the RHS can be constructed again using Berline-Getzler-Vergne.

8. D-branes

We can re-write Wang's main result:

Theorem. (Wang) Given a twisting $\alpha : X \to K(\mathbb{Z}, 3)$ on a smooth closed manifold X, every twisted K-class in $K^{ev/odd}(X, \alpha)$ is represented by a twisted geometric cycle obtained from an $(\alpha + (W_3 \circ \tau))$-twisted-$Spin^c$ closed manifold M and an ordinary K-class $[E] \in K^0(M)$.

and summarise its consequences:

there are three definitions of twisted K-theory $K^*(X,\alpha)$ for a smooth manifold X, all equivalent:

1. A topological definition in terms of homotopy equivalence classes of sections of a bundle of K-theory spectra associated to (X,α).

2. An analytical definition in terms of the continuous trace C^*-algebra associated to (X,α).

3. A geometric definition in terms of a geometric cycle (M,ι,ν,η,E) using a twisted $Spin^c$ structure for $\iota : M \to X$ combined with twisted Poincaré duality.

We claim that this twisted geometric cycle is the so-called Type II D-brane for a class in $K^*(X,\alpha)$. The equivalence of these three definitions gives a candidate for the D-brane charge map on some (to be defined) category of D-branes:

$$\{\text{D-branes over } (X,\alpha)\} \longrightarrow K^*(X,\alpha).$$

Inherent in this supposed categorical point of view is finding a way to capture Bott Periodicity in a more ‘D-brane’ way, than vector bundle modification.

Remark: (Wang) There is a version of Type I D-branes using twisted $Spin$-manifolds over (X,α) with $\alpha : X \to K(\mathbb{Z}_2, 2)$.

9. Remarks on String structures

One may think of the obstruction to the existence of a string structure on the loop space LM as an analogue of the class $W_3(M)$ except that the string class lies in $H^4(M, \mathbb{Z})$.

So the $K(\mathbb{Z}, 3)$-twist suggests we take $K(\mathbb{Z}, 4)$ valued twists as the notion of twisted string structure for a closed manifold X.

Define a string twisting to be given by a smooth map

$$\alpha : X \longrightarrow K(\mathbb{Z}, 4).$$

Arguing by analogy with the earlier theory we arrive at the next definition.

Definition. (Wang) Let (X, α) be a closed manifold with a string twisting $\alpha : X \to K(\mathbb{Z}, 4)$. An α-twisted **string** manifold over X is a quadruple (M, ν, ι, η) where

1. M is a smooth compact manifold with a stable spin structure on its normal bundle given by $\nu : M \longrightarrow \mathbf{BSpin}$ where $\mathbf{BSpin} = \varinjlim_{k} \mathbf{BSpin}(k)$ is the classifying space for stable spin structures;
2. $\iota : M \to X$ is a continuous/smooth map;
3. η is an α-twisted string structure on M, that is a homotopy commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\nu"] \arrow[d, "\iota"'] & \mathbf{BSpin} \arrow[d, "\frac{p_1}{2}"] \arrow[dl, Rightarrow, dashed, "\eta"'] \\
X \arrow[r, "\alpha"'] & K(\mathbb{Z}, 4),
\end{tikzcd} .$$

Here we recall that the string class $\frac{p_1(M)}{2}$ is half the first Pontriagin class of TM.

So in Wang's picture he is letting $\frac{p_1}{2} : \mathbf{BSpin} \to K(\mathbb{Z}, 4)$ be the classifying map of the principal $K(\mathbb{Z}, 3)$-bundle $\mathbf{BString} \to \mathbf{BSpin}$, representing the generator of $H^4(\mathbf{BSpin}, \mathbb{Z})$, and η is a homotopy between $\dfrac{p_1}{2} \circ \nu$ and $\alpha \circ \iota$.

Remark. Arguing by analogy with the previous case we can see that the following holds.

Given a smooth compact spin manifold M and a manifold X with a string twisting $\alpha : X \to K(\mathbb{Z}, 4)$.

1. M admits an α-twisted string structure if and only if there is a continuous map $\iota : M \to X$ such that
$$\iota^*([\alpha]) + \frac{p_1(M)}{2} = 0 \tag{1}$$
in $H^4(M, \mathbb{Z})$.

2. In arXiv:0910.4001 Sati, Schrieber and Stasheff showed that using this notion of twisted string structure one may give a topological interpretation for the Witten quantisation condition in M-theory.

A similar observation was made independently by Matt Ando. The connection with elliptic cohomology is based on his work with Michael Hopkins, and Charles Rezk and results in Matthew Ando, Andrew J. Blumberg, David J. Gepner, Michael J. Hopkins, Charles Rezk. " Units of ring spectra and Thom spectra", arxiv:0810.4535.

The idea is that there must be a connection between the elliptic cohomology of M and that of X. Ando conjectures also a direct application to TMF (work in progress).

SO on to ESI 2012!

Remark on T-duality

Given a principal T^n-bundle $p: Y \to X$ with a twisting α on Y satisfying $p_!\alpha = 0 \in H^1(X, \mathbb{Z}^{\frac{n(n-1)}{2}})$, there is a classical T-dual $(Y^\#, \alpha^\#)$ such that

$$K^*(Y, \delta) \cong K^{*+n}(Y^\#, \delta^\#).$$

The dependence of twisted Chern character

$$ch_{\check{\alpha}} : K^*(Y, \alpha) \longrightarrow H^*(Y, curv(\check{\alpha}))$$

on $\check{\alpha}$ (a gerbe connection and curving) makes the geometric formulation of classical T-duality, in terms of geometric cycles with connection

$$(M, \iota, \nu, \eta, E, \nabla_E),$$

This is unfinished work.