ELLIPTIC GAMMA FUNCTION PROVIDES THE ČECH COCYCLE OF A GERBE

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Abstract. G. Felder and A. Varchenko discovered certain modular formulas for elliptic gamma functions. These identities are generalized to an infinite set of identities for elliptic gamma functions associated to pairs of planes in 3-dimensional space in [2]. There we also use the language of stacks and gerbes to give a natural framework for a systematic description of these identities and their domain of validity. In this note I summarize the work in [2] with an emphasize on the Čech open covers.

The elliptic gamma function [6] is a function of three complex variables obeying

$$\Gamma(z+\sigma,\tau,\sigma) = \theta_0(z,\tau)\Gamma(z,\tau,\sigma),$$
$$\theta_0(z,\tau) = \prod_{j=0}^{\infty}(1-e^{2\pi i((j+1)\tau - z)})(1-e^{2\pi i(j\tau+z)}).$$

In [3] three-term relations for Γ involving $ISL_3(\mathbb{Z}) = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^3$ were discovered, generalizing the modular properties of theta functions under $ISL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.

In [2], we show that these identities are a special case of a set of three-term relations for a family of gamma functions $\Gamma_{a,b}$, which are interpreted geometrically as giving a meromorphic section of a hermitian gerbe on the *universal triptic curve*. The result generalizes the fact that the theta function θ_0 is a section of a hermitian line bundle on the universal elliptic curve. We call this gerbe **gamma gerbe**.

We describe gamma gerbe by the enlarged gamma function family. For a, b linearly independent in the set Λ_{prim} of primitive vectors (namely not multiples of other vectors) in the lattice $\Lambda = \mathbb{Z}^3$, there is a unique primitive $\gamma \in \Lambda^\vee_{prim}$ in the dual lattice such that $\det(a,b,\cdot) = s\gamma$ for some $s > 0$. For $w \in \mathbb{C}, x \in \Lambda \otimes \mathbb{C} = \mathbb{C}^3$ for which the products

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converge we define

$$\Gamma_{a,b}(w,x) := \frac{\prod_{\delta\in C_{+-}/\mathbb{Z}\gamma}(1-e^{-2\pi i(\delta(x)-w)/\gamma(x)})}{\prod_{\delta\in C_{-+}/\mathbb{Z}\gamma}(1-e^{2\pi i(\delta(x)-w)/\gamma(x)})},$$

where $C_{+-} = C_{+-}(a,b) = \{\delta \in \Lambda^\vee | \delta(a) > 0, \delta(b) \leq 0\}$ and $\mathbb{Z}\gamma$ acts on it by translation. We set similarly $C_{-+}(a,b) = C_{+-}(b,a)$. We define $\Gamma_{a,\pm a} = 1$. The function $\Gamma_{a,b}$ is meromorphic on $\mathbb{C}\times(U_a^+ \cap U_b^+)$, where U_a^+'s are calfully chosen as

$$U_a^+ = \{x \in \mathbb{C}^3 | \mathrm{Im}(\alpha(x)\overline{\beta(x)}) > 0\}$$

for any oriented basis α, β of the plane $\delta(a) = 0$. It is easy to check that U_a^+ is independent of the choice of α and β.

For linearly independent $a, b \in \Lambda_{prim}$, $\Gamma_{a,b}$ is a finite product of ordinary elliptic gamma functions:

$$\Gamma_{a,b}(w,x) = \prod_{\delta\in F/\mathbb{Z}\gamma} \Gamma\left(\frac{w+\delta(x)}{\gamma(x)}, \frac{\alpha(x)}{\gamma(x)}, \frac{\beta(x)}{\gamma(x)}\right), \tag{1}$$

for any $\alpha, \beta \in \Lambda^\vee$ satisfying $\alpha(b) = \beta(a) = 0$ and $\alpha(a) > 0$ $\beta(b) > 0$, $F = \{\delta \in \Lambda^\vee | 0 \leq \delta(a) < \alpha(a), 0 \leq \delta(b) < \beta(b)\}$.

Let X be the total space of the line bundle $O(1) \to (\mathbb{C}P^2 - \mathbb{R}P^2)$. Geometrically we think of $O(1)$ as the dual bundle to the tautological line bundle and of $\mathbb{C}P^2$ as the projectivization of $\Lambda_\mathbb{C} = \Lambda \otimes_\mathbb{Z} \mathbb{C}$, where $\Lambda = \mathbb{Z}^3$ is a free abelian group of rank 3 equipped with a volume form $\det : \wedge^3\Lambda \to \mathbb{Z}$. The group $\mathrm{Aut}(\Lambda) \cong SL_3(\mathbb{Z})$ of linear transformations of $\Lambda_\mathbb{C}$ mapping Λ to itself and preserving the volume form, acts naturally on X. The dual lattice $\Lambda^\vee = \mathrm{Hom}(\Lambda, \mathbb{Z}) \cong \mathbb{Z}^3 \subset V_\mathbb{C}^*$ acts on $O(1)$ fiberwise by translation and we get an action of $\mathrm{Aut}(\Lambda) \ltimes \Lambda^\vee \cong ISL_3(\mathbb{Z})$. More explicitly, this group acts linearly on $\mathbb{C} \times V_\mathbb{C}$ via $(g,\mu)(w,x) = (w - \alpha(x), gx)$, and this action induces an action on $X = (\mathbb{C} \times (V_\mathbb{C} - \mathbb{C}\cdot V_\mathbb{R}))/\mathbb{C}^\times$.

The complex manifold X has a natural $ISL_3(\mathbb{Z})$-equivariant open covering $\mathcal{V} = (V_a)_{a\in\Lambda_p}$ labeled by the set of primitive vectors Λ_p in Λ: for $a \in \Lambda_p$, $V_a = \{(w,x) \,|\, x \in U_a^+\}/\mathbb{C}^\times$. Our extended elliptic gamma functions $\Gamma_{a,b}$ will give some Čech cocycle with respect to this $ISL_3(\mathbb{Z})$-equivariant open covering $\mathcal{V} = (V_a)_{a\in\Lambda_p}$. We first prove some properties of V_a's.

Proposition 0.1. *For any $a_1, \ldots, a_p \in \Lambda_p$, $\cap_k V_{a_k}^+$ is either empty or contractible.*

Proof. Let $\tilde{U}_a^+ = U_a^+/\mathbb{C}^\times$. The V_a are trivial line bundles over $\tilde{U}_a^+$ so it is sufficient to prove the claim for $\tilde{U}_a^+$.

We divide the proof into two cases:
Case 1: Suppose there are three linearly independent elements in $\{a_k\}$. Since a $SL_3(\mathbb{Q})$ transformation of $\mathbb{R}^3 \supset \Lambda_p$ will not change the topology of $\cap_k \tilde{U}^+_{a_k}$, and any three linearly independent rational vectors can be transformed to the basis vectors under $SL_3(\mathbb{Q})$, we might as well assume that they are e_1, e_2, e_3. Then $\cap_i \tilde{U}^+_{e_i} = \{[(x_1, x_2, x_3)] : \mathrm{Im}(x_1\bar{x}_2) > 0, \mathrm{Im}(x_2\bar{x}_3) > 0, \mathrm{Im}(x_3\bar{x}_1) > 0\}$. We may assume that $x_3 = 1$, $x_1 = r_1 e^{-i\phi}$ and $x_2 = r_2 e^{i\psi}$ with $r_1, r_2 > 0$. Then the above set is exactly the set where

$$0 < \phi < \pi, \quad 0 < \psi < \pi, \quad 0 < 2\pi - \phi - \psi < \pi, \tag{2}$$

and $r_1, r_2 \in \mathbb{R}^+$. This gives us the constraint on (ϕ, ψ) that it is inside a triangle Δ bounded by three lines defined by the above linear equations in the $\mathbb{R}^2$ and no constraint on r_1 and r_2.

Any other vector a_k can be written as $a_k = s_k e_1 + t_k e_2 + u_k e_3$ with $s_k, t_k, u_k \in \mathbb{Q}$. Assume at first that $u_k \neq 0$. Then a basis in $H(a_k)$ can be chosen as $\alpha = u_k\epsilon_1 - s_k\epsilon_3$ and $\beta = u_k\epsilon_2 - t_k\epsilon_3$. Then $\tilde{U}^+_{a_k}$ gives us the constraint that

$$\mathrm{Im}((u_k x_1 - s_k x_3)(u_k\bar{x}_2 - t_k\bar{x}_3)) > \text{or} < 0,$$

depending on the orientation of α and β. This inequality is equivalent to

$$\sin(\psi)s_k\rho_1 + \sin(\phi)t_k\rho_2 - \sin(\psi+\phi)u_k > \text{or} < 0, \tag{3}$$

where $\rho_i = r_i^{-1}$. Notice the symmetry of this inequality. In fact, we will arrive at the same inequality if we assume $s_k \neq 0$ or $t_k \neq 0$. Therefore for a fixed value of (ϕ_0, ψ_0), the restriction of ρ_i's is given by a series of linear equations on $\mathbb{R}^2$. Hence ρ_i's have to be in a certain polygon P in $\mathbb{R}^2$.
With the condition (2), we observe that $\left(\frac{\sin\phi\sin(\phi_0+\psi_0)}{\sin\phi_0\sin(\phi+\psi)}\rho_1, \frac{\sin\psi\sin(\phi_0+\psi_0)}{\sin\psi_0\sin(\phi+\psi)}\rho_2\right)$ satisfies (3) at point (ϕ, ψ) as long as (ρ_1, ρ_2) satisfies (3) at the point (ϕ_0, ψ_0).

Then it is easy to see that the following map $\cap_k \tilde{U}^+_{a_k} \to \Delta \times P$ is a homeomorphism:

$$[(\rho_1^{-1}e^{-i\phi}, \rho_2^{-1}e^{i\psi}, 1)] \mapsto (\phi, \psi, \frac{\sin\phi\sin(\phi_0+\psi_0)}{\sin\phi_0\sin(\phi+\psi)}\rho_1, \frac{\sin\psi\sin(\phi_0+\psi_0)}{\sin\psi_0\sin(\phi+\psi)}\rho_2).$$

Case 2: Suppose that all a_k's lie in the same plane. Since $\tilde{U}^+_a$ is homeomorphic to $\tilde{U}^+_{e_1}$ which is $\mathbb{C} \times H_+$ where H_+ is the upper half plane, the claim is trivial in the case that all a_k lie on the same line. By [2, Lemma 3.9] the intersection is empty if all of a_k's do not lie on the same side of some plane. So after a $SL_3(\mathbb{Q})$ transformation, we can assume $a_1 = e_1$ and $a_2 = e_2$ and everything else lies in between, namely $a_k = s_k e_1 + t_k e_2$ with $s_k, t_k \in \mathbb{Q}^+$. $\tilde{U}^+_{e_1} \cap \tilde{U}^+_{e_2}$ consists of points such that $\mathrm{Im}(x_1 \bar{x}_3) < 0$ and $\mathrm{Im}(x_2 \bar{x}_3) > 0$. After normalizing x_3 to 1, $\tilde{U}^+_{e_1} \cap \tilde{U}^+_{e_2}$ is simply $H_+ \times H_-$. On the other hand, an oriented basis of the plane $H(a_k) := \{\alpha(a) = 0\} \subset \Lambda^\vee$ can be chosen as $s_k \epsilon_2 - t_k \epsilon_1$ and ϵ_3. So $\mathrm{Im}(s_k x_2 - t_k x_1)\bar{x}_3 > 0$ always holds, namely $\tilde{U}_{a_k} \subset \tilde{U}^+_{e_1} \cap \tilde{U}^+_{e_2}$. Therefore, in this case, $\cap \tilde{U}_{a_k} = \tilde{U}^+_{e_1} \cap \tilde{U}^+_{e_2} = H_+ \times H_-$ is contractible. □

Proposition 0.2. *For any $a_1, \ldots, a_p \in \Lambda_p$, $\cap_k V^+_{a_k}$ is either empty or Stein.*

Proof. Submanifolds of $\mathbb{C}^n$ given by $|f_j(z)| < 1$, for $f_1, ..., f_k$ holomorphic are *domains of holomorphy* [5, Theorem 2.5.13]. Domains of holomorphy are Stein manifolds. The open sets V_a are isomorphic as complex manifolds to $H \times \mathbb{C} \times \mathbb{C}$, where H is the upper half plane. Hence V_a is a domain of holomorphiy with $k = 1$, $f_1(z) = e^{iz_1}$. By [5, Cor. 2.5.7] finite intersections of domains of homolorphy are domains of holomorphy again. □

Analytic coherent sheaves (such as $\mathcal{O}$) on Stein manifolds have vanishing cohomology in positive degree [5, Theorem 7.4.3]. If we have a contractible Stein manifold we reach the same conclusion for $\mathcal{O}^\times$ with the long exact sequence. Hence our covering $\mathcal{V}$ is a good covering in the sense that $H^{>0}(\cap_k V^+_{a_k}, \mathcal{O}^\times) = 0$. This implies that the Čech cohomology with respect to $\mathcal{V}$

$$\check{H}^\bullet_{\mathcal{V}}(\mathcal{X}, \mathcal{O}^\times) = H^\bullet(\mathcal{X}, \mathcal{O}^\times),$$

calculate the sheaf cohomology of $\mathcal{X}$.

The functions $\Gamma_{a,b}$ satisfy cocycle conditions generalizing the three-term relations of [3]:

$$\Gamma_{a,b}(w,x)\Gamma_{b,a}(w,x) = 1, \quad x \in U^+_a \cap U^+_b, \tag{4}$$

$$\Gamma_{a,b}(w,x)\Gamma_{b,c}(w,x)\Gamma_{c,a}(w,x) = \exp\left(-\frac{\pi i}{3} P_{a,b,c}(w,x)\right), \quad x \in U^+_a \cap U^+_b \cap U^+_c,$$

where $P_{a,b,c}(w,x) \in \mathbb{Q}(x)[w]$ can be explicitly described in terms of the Bernoulli polynomial $B_{3,3}$, see [2]. Moreover the gamma functions obey cocycle identities related to the action of the group $ISL_3(\mathbb{Z}) = SL_3(\mathbb{Z}) \ltimes \mathbb{Z}^3$. Fix a *framing* of Λ_{prim}, namely for each $a \in \Lambda_{prim}$ a choice of oriented basis $(\alpha_1, \alpha_2, \alpha_3)$ of $\Lambda^\vee \otimes \mathbb{R}$ such that $\alpha_1(a) = 1$,

$\alpha_2(a) = \alpha_3(a) = 0$. Let

$$\Delta_a((g,\mu);w,x) = \prod_{j=0}^{\mu(g^{-1}a)-1} \theta_0\left(\frac{w+j\alpha_1(x)}{\alpha_3(x)}, \frac{\alpha_2(x)}{\alpha_3(x)}\right),$$

where $(g,\mu) \in ISL_3(\mathbb{Z}) = SL_3(\mathbb{Z}) \ltimes \mathbb{Z}^3$. Then we have

$$\frac{\Gamma_{a,b}(w+\mu(g^{-1}x),x)}{\Gamma_{a,b}(w,x)} = e^{\pi i P_{a,b}((g,\mu);w,x)} \frac{\Delta_a((g,\mu);w,x)}{\Delta_b((g,\mu);w,x)}, \tag{5}$$

$$\Delta_a(\hat{g}\hat{h};w,x) = e^{2\pi i P_a(\hat{g},\hat{h};w,x)} \Delta_a(\hat{g};w,x)\Delta_a(\hat{h};w-\mu(x),gx), \tag{6}$$

where $\hat{g} = (g,\mu), \hat{h} = (h,\nu)$ and $P_{a,b}$, P_a are again in $\mathbb{Q}(x)[w]$.

By equations (4) (5) (6) we have:

Theorem 0.3. *There is an $ISL_3(\mathbb{Z})$-equivariant Čech 2-cocycle*

$$(\phi_{a,b,c},\phi_{a,b},\phi_a) = (e^{-\frac{2\pi i}{3!}P_{a,b,c}(w,x)}, e^{-\frac{2\pi i}{2!}P_{a,b}((g,\mu);w,x)}, e^{-2\pi i P_a((g,\mu),(h,\nu);w,x)}),$$

in $C^2_{ISL_3(\mathbb{Z})}(\mathcal{V},\mathcal{O}^\times)$. The image of ϕ in the equivariant Čech complex with values in the sheaf $\mathcal{M}^\times$ of invertible meromorphic sections is the coboundary of the equivariant cochain $(\Gamma_{a,b},\Delta_a) \in C^1_{ISL_3(\mathbb{Z})}(\mathcal{V},\mathcal{M}^\times)$.

The **gamma gerbe** $\mathcal{G}$ is the holomorphic equivariant gerbe on X corresponding to ϕ. Equivalently, it is a holomorphic gerbe on the stack $\mathcal{X} = [X/ISL_3(\mathbb{Z})]$.

More geometrically, if we view gerbes over stacks as central extensions of groupoids, then $\mathcal{G}$ is presented by a groupoid $R \rightrightarrows U_0$ fitting in the central extension of groupoids over U_0:

$$1 \to \mathbb{C}^\times \times U_0 \to R \to U_1 \to 1,$$

where $U_0 = \sqcup V_a$, $U_1 = U_0 \times_X \times (ISL_3(\mathbb{Z}) \times X) \times_X U_0$, $R = \sqcup L_{a,b} \otimes L_b(g)^{-1}$ with $L_{a,b}$, $L_b(g)$ the $\mathbb{C}^\times$-bundles with transition functions $\phi_{a,b,c}\phi^{-1}_{a,b,d}$ (on $(V_a \cap V_b) \cap V_c \cap V_d$) and $\phi_{b,b'}(g,-)\phi^{-1}_{b,b''}(g,-)$ (on $V_b \cap V_{b'} \cap V_{b''}$) respectively. Notice that $U_1 = \cup W_{g,a,g^{-1}b}$ where $W_{g,a,g^{-1}b} = \{(g,y) | y \in V_a, g^{-1}y \in V_{g^{-1}b}\}$. Then $\Gamma_{a,b}\Delta_b^{-1}$ provides a meromorphic groupoid homomorphism $U_1 \to R$, hence Γ's and Δ's can be viewed as a meromorphic section of $\mathcal{G}$. A *hermitian structure of a gerbe* in this language is simply a hermitian structure of the complex line bundle associated to the central extension. There is a *unique* compatible connective on a hermitian gerbe and its curvature represents the Dixmier-Douady class of the gerbe.

Theorem 0.4. *Using the notation in* (1)*, there is a hermitian structure $h_{a,b}h_b^{-1}$ on $\mathcal{G}$ with $h_{a,b}(w,x) = \prod_{\delta\in F/\mathbb{Z}\gamma} h_3\left(\frac{w+\delta(x)}{\gamma(x)}, \frac{\alpha(x)}{\gamma(x)}, \frac{\beta(x)}{\gamma(x)}\right)$ and*

$h_a((g,\mu);w,x) = \prod_{j=0}^{\mu(g^{-1}a)-1} h_2\left(\frac{w+j\alpha_1(x)}{\alpha_3(x)}, \frac{\alpha_2(x)}{\alpha_3(x)}\right)$, *where* h_n *are defined by Bernoulli polynomials:*

$$h_n(z,\tau_1,\ldots,\tau_{n-1}) = \exp\left(-(4\pi/n!)B_{n-1,n}(\zeta,t_1,\ldots,t_{n-1})\right),$$

with $\zeta = \operatorname{Im} z$, $t_j = \operatorname{Im}\tau_j$.

Moreover, as with line bundles, we can construct the gamma gerbe $\mathcal{G}$ via (pseudo)-divisors. A *triptic curve* $\mathcal{E}$ is a holomorphic stack of the form $[\mathbb{C}/\iota(\mathbb{Z}^3)]$ with $\iota : \mathbb{Z}^3 \to \mathbb{C}$ a map of rank 2 over $\mathbb{R}$. An *orientation* of a triptic curve $\mathcal{E}$ is given by a choice of a generator of $H^3(\mathcal{E},\mathbb{Z}) \cong \mathbb{Z}$. Then the stack $\mathcal{T}r := [(\mathbb{C}P^2 - \mathbb{R}P^2)/SL_3(\mathbb{Z})]$ is the moduli space of oriented triptic curves. The stack $\mathcal{X} = [X/ISL_3(\mathbb{Z})]$ is the total space of the universal family of triptic curves over $\mathcal{T}r$. Given an étale map $U \to \mathcal{E}$, let $Z_U = 0 \times_{\mathcal{E}} U$. Z_U is naturally a discrete subset of a principal oriented $\mathbb{R}$-bundle on U. Then a *pseudodivisor* on U is a function $D : Z_U \to \mathbb{Z}$ such that if $\lim y_n = +\infty$ (resp. $-\infty$) for a sequence y_n in Z_U with compact support in U then $\lim D(y_n) = 1$ (resp. -1). The notion of positive/negative infinity is derived from the orientation of the fibres of the $\mathbb{R}$-bundle. We can globalize this to $\mathcal{X}$, namely for an étale map $U \to \mathcal{X}$, a *pseudodivisor* on U is a function $D : \mathcal{T}r \times_{\mathcal{X}} U \to \mathbb{Z}$ such that for every point $q \to \mathcal{T}r$ with corresponding fibre $\mathcal{E} = q \times_{\mathcal{T}r} \mathcal{X}$, the restriction $q \times_{\mathcal{X}} U$ is a pseudodivisor on $U \times_{\mathcal{X}} \mathcal{E}$. Then for two such D_i's, the pushforward $p_*(D_1 - D_2)$ is a divisor on U, hence can be used to twist a line bundle L to $L(p_*(D_1 - D_2))$, where $p : \mathcal{T}r \times_{\mathcal{X}} U \to U$. Using the categorical description of gerbes in [1], we then have

Theorem 0.5. *The gamma gerbe* $\mathcal{G}$ *is a gerbe over* $\mathcal{X}$ *made up by the following data: for* U *with an étale open map* $U \to \mathcal{X}$,

$$Obj(\mathcal{G}_U) = \{(L,D) |\ L \text{ is a line bundle on } U \text{ and } D \text{ a pseudodivisor}\},$$
$$Mor(\mathcal{G}_U)((L_1,D_1) \to (L_2,D_2)) = \Gamma^\times(U,\ \big(L_1^* \otimes L_2\big)(p_*(D_2 - D_1)),$$

the invertible holomorphic sections.

We also have the following theorems calculating various cohomology groups and Diximer-Douady classes of the gamma gerbes or its restriction.

Theorem 0.6. *Let* $\mathcal{E}_r = \mathbb{C}/\iota(\mathbb{Z}^r)$, *where* $x_j = \iota(e_j)$, *the images of the standard basis vectors, are assumed to be* $\mathbb{Q}$*-linearly independent and to span* $\mathbb{C}$ *over* $\mathbb{R}$. *Then*

$H^{i\le r-2}(\mathcal{E}_r,\mathcal{O}^\times) = \wedge^i(\mathbb{C}^r/(x_1,...,x_r)\mathbb{C})/\wedge^i(\mathbb{Z}^r)$, $H^{r-1}(\mathcal{E}_r,\mathcal{O}^\times) = \mathcal{E}_r \times \mathbb{Z}$, *and* $H^{\geq r}(\mathcal{E}_r, \mathcal{O}^\times) = 0$. *In particular, for generic triptic curves* $\mathcal{E}$*, the groups classifying holomorphic and topological gerbes on* $\mathcal{E}$ *are*

$$H^2(\mathcal{E}, \mathcal{O}^\times) = \mathcal{E} \times \mathbb{Z} \quad \textit{and} \quad H^3(\mathcal{E}, \mathbb{Z}) = \mathbb{Z},$$

respectively.

Theorem 0.7. *The Dixmier-Douady class* $c(\mathcal{G}|_{\mathcal{E}})$ *of the restriction of the gamma gerbe to* $\mathcal{E}$ *is a generator of* $H^3(\mathcal{E}, \mathbb{Z}) = \mathbb{Z}$.

Theorem 0.8. $H^3(\mathcal{X}, \mathbb{Z})$ *fits in a short exact sequence*

$$0 \to \mathbb{Z} \to H^3(\mathcal{X}, \mathbb{Z})/\textit{torsion} \to H^3(\mathbb{Z}^3, \mathbb{Z}) \cong \mathbb{Z} \to 0.$$

The image of the Dixmier-Douady class $c(\mathcal{G}) \in H^3(\mathcal{X}, \mathbb{Z})$ *of the gamma gerbe is a generator of* $H^3(\mathbb{Z}^3, \mathbb{Z})$.

There should exist non-abelian versions of this story in the context of q-deformed conformal field theory [4].

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