Torsion in the homology of locally symmetric spaces

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Introduction

Let G be a real semisimple Lie group, $K \subset G$ a maximal compact subgroup and $\Gamma \subset G$ a lattice.

De George-Wallach, Clozel, Savin, Delorme, Lück ... have studied the asymptotic behavior of the homology groups $H_j(\Gamma, \mathbb{C})$ when Γ varies - for instance, shrinks to $\{1\}$ through a family of subgroups of a fixed lattice.

In particular if $\Gamma_N \subset \Gamma$ is family of normal subgroups with $\cap_N \Gamma_N = \{1\}$, the limit

$$\lim_{N \to +\infty} \frac{\dim H_j(\Gamma_N, \mathbb{C})}{[\Gamma : \Gamma_N]}$$

is known to exist for every j and is non zero if and only if

$$\delta(G) := \mathrm{rank}_{\mathbb{C}}(G) - \mathrm{rank}_{\mathbb{C}}(K) = 0 \quad \text{and} \quad j = \frac{1}{2}\dim(G/K).$$

Examples:

- $\delta = 0$ for the groups $\mathrm{SU}_{p,q}$, $\mathrm{SO}_{p,q}$ (pq even);
- $\delta = 1$ for the groups $\mathrm{SL}_3(\mathbb{R})$, $\mathrm{SL}_4(\mathbb{R})$, $\mathrm{SO}_{p,q}$ (pq odd);
- $\delta = 2$ for the groups $\mathrm{SL}_5(\mathbb{R})$, E_6^{split}.

In our work we investigate the corresponding question where Betti numbers are replaced by the size of torsion components of homology groups.

We were first motivated by the case $G = \mathrm{SL}_2(\mathbb{C})$. Locally symmetric spaces associated to $\mathrm{SL}_2(\mathbb{C})$ are hyperbolic 3-manifolds.

Elstrodt, Grunewald, Mennicke, Cremona, Figueiredo : Tabulation of Γ^{ab} for Γ congruence subgroups of small level.

"*These tables show that the groups Γ^{ab} can have big torsion subgroups. We do not have any general result on the nature of these torsion elements.*"

Why study torsion classes ?

According to standard conjectures (belonging to Langlands' general philosophy and made precise by Serre, Ash, ...): torsion homology (Hecke eigen-)classes should correspond to certain (odd) Galois representations (with a specified Hodge-Tate weight).

Recent works of Calegari and Mazur on the Galois side suggest that indeed arithmetic hyperbolic 3-manifolds should have a lot of torsion in their homology.
This urges for some general result on the geometric side !

Theorem (Silver-Williams, 2002)

Let $k \subset \mathbb{S}^3$ be a hyperbolic knot. Given a positive integer N we let M_N be the N-th cyclic cover of $\mathbb{S}^3$ ramified over k. Then:

$$\lim_{N \to +\infty} \frac{\log |H_1(M_N, \mathbb{Z})_{\mathrm{tors}}|}{N} > 0.$$

A conjecture

Jointly with Akshay Venkatesh we study the amount of torsion in the homology of a general arithmetic locally symmetric space. We give, in particular, examples where the size of the torsion group grows exponentially with the covolume and formulate conjectures on when this will happen in general.

Coarsely speaking, we believe that:

Torsion is “large” exactly in the case when $\delta(G) = 1$.

Indeed $\delta(\mathrm{SL}_2(\mathbb{C})) = 1$.

Let G be an anisotropic semisimple $\mathbb{Q}$-group and $\Gamma \subset G(\mathbb{Q})$ be a congruence subgroup.
Let $F/\mathbb{Q}$ be a finite extension and $\mathcal{O}_F$ its ring of integers.
Fix W a finite dimensional representation of G on F and $M \subset W$ a Γ-invariant $\mathcal{O}_F$-lattice in W.

Given $\Gamma_N \subset \Gamma$ a decreasing sequence of congruence subgroups with the property that $\cap_N \Gamma_N = \{1\}$ we may consider the homology groups

$$H_j(\Gamma_N, M)$$

with values in the local system determined by M. These are finitely generated $\mathbb{Z}$-modules with a free part and a torsion part.

Conjecture (B. - Venkatesh, 2009)

The limit

$$\lim_{N\to+\infty} \frac{\log|H_j(\Gamma_N, M)_{\mathrm{tors}}|}{[\Gamma : \Gamma_N]}$$

exists for each j and is zero unless

$$\delta(G) = 1 \quad \text{and} \quad j = \frac{1}{2}(\dim(G/K) - 1).$$

In that last case, it is always positive and equal to an explicit constant $c_{G,W}$ times the volume of $\Gamma\backslash G/K$.

This speculatice conjecture contains in fact 3 statements whose origin are of different nature:

• If $\delta = 0$, then there is little torsion whereas $H_j(\mathbb{Q})$ is large; the torsion is almost entirely "absorbed" by the characteristic zero homology.

• If $\delta = 1$, then there is "a lot" of torsion but $H_j(\mathbb{Q})$ is small.

• If $\delta \geq 2$, there is "relatively little" torsion or characteristic zero homology.

Galois heuristics

Heuristic I

The expected number (as K varies) of Galois representations mod. p of "Ash-Serre type" is:

$$\approx p^{-\delta}(1 + O(p^{-1})).$$

Heuristic II

The chance that a specified torsion class mod. p lifts to a mod. p^2 class is

$$\approx p^{-\delta}.$$

Part of the heuristic can be made rigorous in the context of Galois deformation rings. It essentially follows the lines of recent works of Calegari and Mazur.

Note that

$$\sum_p p^{-m} = \begin{cases} \infty & \text{if } m = 1 \\ \text{finite} & \text{if } m > 1. \end{cases}$$

Heuristic I: if $\delta = 0$ or 1 there is a lot of torsion mod. p.
Heuristic II: if $\delta > 0$ most of the mod. p torsion classes don't lift mod. p^2.

In brief: there should be a lot of torsion in the homology exactly when $\delta = 1$!

A theorem

In support of the "large torsion" direction, we show:

Theorem (B. - Venkatesh, 2009)

Suppose that $\delta(G) = 1$ and that the Γ-module M is strongly acyclic. Then:

$$\liminf_{N\to+\infty} \frac{1}{[\Gamma : \Gamma_N]} \sum_{j=1}^{\dim(G/K)} \log |H_j(\Gamma_N, M)_{\mathrm{tors}}| \geq c_{G,W} \mathrm{vol}(\Gamma\backslash G/K) > 0.$$

The term "strongly acyclic" means that $M \otimes \mathbb{R}$ has no small eigenvalues. It only depends on the real representation.

Example: The real representation

$$\mathrm{Sym}^p(\mathbb{C}^2) \otimes \overline{\mathrm{Sym}^q(\mathbb{C}^2)}$$

of $\mathrm{SL}_2(\mathbb{C})$ is strongly acyclic if and only if $p \neq q$.

In general we cannot isolate the degree which produces torsion *except* in certain low degree examples. For example:

If $G = \mathrm{SL}_2(\mathbb{C})$ (resp. $\mathrm{SL}_3(\mathbb{R})$) then the corresponding result holds for H_1 (resp. H_2) with $\liminf$ replaced by $\lim$ and $\geq$ be equality.

In these cases, then, our theorem amounts to a verification for (most) twisted-coefficients of our conjecture.

An explicit corollary

Let F be a numberfield with exactly two complex (conjugate) embeddings and B/F a quaternion algebra. Suppose that B is unramified at the complex places and ramifies at all the real places. Then $\mathbf{G} = B^1$ is an algebraic group over F and $G = \mathbf{G}(F \otimes \mathbb{R})$ is a product of $\mathrm{SL}_2(\mathbb{C})$ by a compact group with associated symmetric space $\mathbb{H}^3$ – the 3-dimensional hyperbolic space.
Let E/F be a finite extension of degree d, such that $B \otimes_F E \cong M_2(E)$. Considering the canonical 2-dimension representation of $M_2(E)$ we get a 2-dimensional representation (V, ρ) of $\mathbf{G}$ over E.

Denote by $\rho_{p,q}$ the representation

$$\mathrm{Sym}^p \rho \otimes \overline{\mathrm{Sym}^q \rho} \quad \text{on} \quad \mathrm{Sym}^p V \otimes \overline{\mathrm{Sym}^q V} \quad (p, q \in \mathbb{N}).$$

Any congruence lattice $\Gamma \subset \mathbf{G}(F)$ may be obtained as the set of norm one elements $\Gamma = \Omega^1$ in an order $\Omega \subset B$.
Let $\mathcal{O}_E$ be the ring of integers of E. Choosing a Γ-invariant lattice in V yields local systems of free finite rank $\mathcal{O}_E$-modules $M_{p,q}$ on $X = \Gamma \backslash \mathbb{H}^3$ associated to each representation $\rho_{p,q}$.

Let $\Gamma_N \subset \Gamma$ be a decreasing sequence of finite index subgroups such that $\cap_N \Gamma_N = \{1\}$.

Corollary

Suppose $p > q$. Then the homology groups $H_j(\Gamma_N, M_{p,q})$ are trivial unless $j = 1$. In that case it is always finite and the limit

$$\lim_{N\to+\infty} \frac{\log |H_1(\Gamma_N, M_{p,q})|}{[\Gamma:\Gamma_N]} = d(2q+1)((2p+1)^2 + 2(p+q+1)(p-q))\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{3\pi}.$$

Some ideas of the proof

The deep component in the proof of the theorem is not due to us; it is a remarkable result of Müller, a generalisation of the "Cheeger-Müller theorem". This result states, loosely speaking, that the size of torsion groups can be computed by analytic methods. This involves the analytic torsion τ_N of the compact quotient $\Gamma_N \backslash G/K$.

Beyond this result, the other techniques are also not original and have been used in other contexts. First, using heat kernel methods (in the spirit of Lott) we prove that the sequence $\frac{\tau_N}{[\Gamma:\Gamma_N]}$ converges toward the so-called L^2-analytic torsion. This is were we need the "strong acyclicity" hypothesis.

Finally, we compute the L^2-analytic torsion (with any twisted coefficient system) of all symmetric spaces G/K. In doing so we generalize previous results of Lott, Hess, Schick and Olbricht. Our slightly different method allows us to verify that these invariants are non-zero if (and only if) $\delta(G) = 1$.

A simple example

Rather than to give more technical details I prefer to describe a very simple example where most of the above features appear.

Let $X = \mathbb{R}/\mathbb{Z}$ be the Euclidean circle and M the local system on X with fiber $\mathbb{Z}^m$ and monodromy $A \in \mathrm{SL}_m(\mathbb{Z})$.
Suppose that A is semisimple and does not admit 1 as an eigenvalue. The cohomology is then torsion and concentrated in H^1:

$$|H^1(X; M)| = |\det(1 - A)|.$$

Let us now compute the de Rham complex:

$$\begin{array}{ccccccc} 0 \to & \begin{array}{c}\Omega^0(X; M_{\mathbb{R}}) \\ f \quad \text{s.t.} \\ f(x+1) = Af(x)\end{array} & \xrightarrow{d} & \begin{array}{c}\Omega^1(X; M_{\mathbb{R}}) \\ f(x)dx \quad \text{s.t.} \\ f(x+1) = Af(x)\end{array} & \to 0 \\ & \downarrow \times e^{-2\pi ixB} & & \downarrow \times e^{-2\pi ixB} & \\ 0 \to & \{f \ : \ f(x+1) = f(x)\} & \xrightarrow{\frac{d}{dx}+2i\pi B} & \{f \ : \ f(x+1) = f(x)\} & \to 0 \end{array}$$

where $\exp(2\pi iB) = A$.

Formally speaking the product of singular values of d may be expressed as:

$$\det{}'(d) := \prod_n \prod_j |2\pi n + 2\pi\lambda_j|$$

where the λ_j's are the eigenvalues of B.

If we compute formally using the identity

$$\prod_{n\geq 1}\left(1-\frac{x^2}{n^2}\right)=\frac{\sin\pi x}{\pi x},$$

we arrive at:

$$\det{}'(d)=|\det(1-A)|\left(\prod_{n\geq 1}2\pi n\right)^2.$$

Thus, in a reasonable method of regularizing products in which $\prod_{n\geq 1}2\pi n=1$, we indeed see that the de Rham complex computes the torsion. The generalisation of this to general Riemannian manifolds is the content of the Cheeger-Müller theorem.

Let now $X \xrightarrow{\pi_N} X$ be the N-fold covering and suppose that A has no eigenvalue that is a root of unity. Then:

$$|H^1(X; \pi_N^* M)| = |\det(1 - A^N)|.$$

If A has no eigenvalues of modulus one, we may interpret the logarithm of

$$|\det(1 - A^N)| = \prod_{\omega^N = 1} |\det(\omega - A)|$$

as a Riemann sum on the circle to get

$$\frac{\log |H^1(X; \pi_N^* M)|}{N} \longrightarrow \log M(\chi_A) := \int_{|z|=1} \log |\chi_A(z)|. \qquad (1)$$

Here M is the *Mahler measure*.

The limit (1) is still true in general, it is equivalent to a non-trivial result (due to Gelfond) on Diophantine approximation: if α is an algebraic number of absolute value 1, then

$$\frac{\log|\alpha^N - 1|}{N} \to 0, \quad N \to +\infty.$$

Let us now examine (1) from the “de Rham” perspective. Carrying out the computations as above, we arrive at:

$$\det{}'(d_N) = \prod_n \prod_j \left|\frac{2\pi n}{N} + 2\pi\lambda_j\right|.$$

Now as $N \to +\infty$, the sequence $\frac{2\pi n}{N} + 2\pi\lambda_j$ fills out densely the line $2\pi\lambda_j + \mathbb{R}$; the "density" is $N/2\pi$. It is therefore reasonable to imagine that:

$$\frac{\log|\det{}'(d_N)|}{N/2\pi} \to \sum_j \int \log|2\pi\lambda_j + x|dx, \qquad (2)$$

where the latter integral is to be understood, again, in a regularized fashion.

It is not hard to check that (2) implies (1). Of course the limit (2) is not easy to check. The issue here is that some λ_j may have

$$\begin{aligned} \mathrm{Im}(\lambda_j) = 0 \quad &\Leftrightarrow \quad A \text{ has an eigenvalue of modulus } 1 \\ &\Leftrightarrow \quad \text{some eigenvalues of } d_N \quad \to 0 \\ &\Leftrightarrow \quad M_{\mathbb{R}} \text{ is not strongly acyclic.} \end{aligned}$$

A contrario if $M_{\mathbb{R}}$ is strongly acyclic, it is not hard to justify (2). Now

$$\sum_j \int \log |2\pi\lambda_j + x| dx = \text{“sum of log” of the } L^2\text{-spectrum of } \widetilde{X} = \mathbb{R} \text{ twisted by } M_{\mathbb{R}}.$$

This last term is the L^2-analytic torsion.

Conclusion

Let us finally return to the case of a locally symmetric manifold $\Gamma\backslash G/K$.
If M is a strongly acyclic local system, we similarly get:

$$\frac{1}{[\Gamma : \Gamma_N]} \sum_i (-1)^i \log |H_i(\Gamma_N, M)| \to T^{(2)}(\Gamma; M_{\mathbb{R}}),$$

where $T^{(2)}(\Gamma; M_{\mathbb{R}})$ – the analytic L^2-torsion twisted by $M_{\mathbb{R}}$ – may be explicitely computed using Harish-Chandra's Plancherel formula.

This conclues the proof ... and the talk !