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Quaternionic Contact Geometries

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M^{4n+3} manifold; $H^{4n} \leq T^{4n+3}(M)$ - distribution;

Dfn: We say that H is a QC (Quaternionic Contact) distribution (or structure) on M if there exist:

- $\eta = (\eta_1, \eta_2, \eta_3) : TM \to \mathbb{R}^3$, $ker(\eta) = H$
- $g : \odot^2 H \to \mathbb{R}$ positive-definite metric on H
- $I_1, I_2, I_3 \in End(H)$,

$$(I_1)^2 = (I_2)^2 = (I_3)^2 = -1, \quad I_1 \circ I_2 = -I_2 \circ I_1 = I_3$$

all satisfying the equations

$$d\eta_s(X, Y) = g(I_s X, Y), \; X, Y \in H, \; s = 1, 2, 3.$$

If H is a QC-distribution, then the equations

$$d\eta_s(X,Y) = g(I_s X, Y), \ s = 1,2,3$$

determine uniquely

- the quaternionic structure $Q = span\{I_1, I_2, I_3\} \in End(H)$
- the conformal class $[g]$ of the metric g on H
- the form $\eta = (\eta_1, \eta_2, \eta_3)$ up to an action of the group $R^+SO(3)$ in $\mathbb{R}^3$.

An important set of examples is given by the 3-Sasakian mfds.

Recall: Riemannian mfd (M,h) is called 3-Sasaki if it admits 3 unit Killing vectorfields ξ_1, ξ_2, ξ_3 satisfying the equations

$$(\nabla^{LC}_X(\nabla^{LC}\xi_i))Y = g(\xi_i, Y)X - g(X,Y)\xi_i,$$

or equivalently, if its metric cone is a hyper-Kähler mfd.

- The distribution $H = \{\xi_1, \xi_2, \xi_3\}^\perp$ is easily seen to be QC.

Motivation: QK Geometry and the inverse twistor construction

Inverse twistor construction [Claude LeBrun 1989] :

• Consider: Z^{2n+1} complex mfd; Θ - holomorphic contact form; σ - antiholomorphic involution.

• We say that $C \in Z$ is a twistor line, if C is a complex compact curve of genus-0 with normal bundle $N(C) \cong \mathcal{O}(1) \otimes \mathbb{C}^{2n}$.

• The set X^c of all twistor lines is a complex mfd whose tangent space at C is $H^0(C, N(C)) \cong \mathbb{C}^{4n}$ [by applying a theorem of Kodaira 1962]

• The set M^c of all twistor lines tangent to $ker(\Theta)$ is a non singular, closed complex hypersurface in X^c.

• The real slice X of X^c given by σ is a Quaternionic mfd. There is a natural quaternionic Kähler metric on $X^c - M^c$ which has pole of order 2 on M^c.

• The real slice M of M^c has a natural QC structure. [Olivier Biquard 2000]

• M is called the conformal infinity of the QK metric on $X - M$.

• The germ of the QK metric on $X - M$ is uniquely determined by the QC structure on M [Olivier Biquard 2000].

• Each analytic QC-mfd could be locally imbedded into a quaternionic mfd as a hypersurface [David Duchemin 2006].

Thm [LeBrun 1991]: The moduli space of complete QK metrics (with negative scalar curvature) on the ball $B \subset R^{4n+4}$ is infinite-dimensional. Hence the moduli space of QC-distributions on the sphere S^{4n+3} is infinite-dimensional as well.

An Example : The Quaternionic Heisenberg group

- $\mathbb{H}$: the non-commutative field of quaternions
- The Heisenberg group of dimension $4n+3$

$$G(\mathbb{H}) := \mathbb{H}^n \times Im(\mathbb{H})$$

- The group law:

$$,(q_1, w_1).(q_2, w_2) = (q_1 + q_2, w_1 + w_2 - 2Im(\bar{q_1}^t q_2))$$

- In the point $0 \in G(\mathbb{H})$ we have the splitting

$$T_0 G(\mathbb{H}) = \mathbb{H}^n + Im(\mathbb{H}).$$

Let H be the left-invariant $4n$-dimensional distribution which coincides with $\mathbb{H}^n$ in 0. Then H is an example of a QC-distribution.

Let H be a QC-distribution on the mfd M.

Thm [Olivier Biquard 2000]: For any $g \in [g]$ there exist a unique (intrinsic) connection ∇ on (M, H) and a distribution V s.t. :

- ∇ preserves $H \oplus V$, g and Q
- ∇ preserves $V \to Q \subset End(H)$

} $Hol^{\nabla} \subset Sp(n)Sp(1)$

- $T(X, Y) = [X, Y]_V$
- $T(\xi, .) \in (sp(n) \oplus sp(1))^{\perp} \subset End(H)$

} intrinsic torsion

where $X, Y \in H, \quad \xi \in V$

Note: For the Heisenberg group $G(\mathbb{H})$, ∇ is exactly the left-invariant (flat) connection.

Thm [Ivanov, Minchev, Vassilev 2007] : If the curvature R^{∇} is zero then (M, g, H) is locally isomorphic to $G(\mathbb{H})$.

Describing the integrable case of $T(\xi, .) = 0$

Define $S := \sum_{s,k} R(e_s, e_k, e_k, e_s)$, where e_s is ONB for H.

- S is called QC-scalar curvature, and it is a constant if g is integrable.

Assume $g \in [g]$ is integrable

and consider the induced splitting

$$TM = H \oplus V^g \quad \Rightarrow \quad g^\lambda, \lambda \neq 0 - metric\ tensors.$$

- g^λ is positive definite if $\lambda > 0$ and with signature $(+4n, -3)$ if $\lambda < 0$.

Thm **[Ivanov, Minchev, Vassilev : work in progress] :** $\exists\, K_1, K_2, K_3$ - Killing vector fields for all g^λ.

Moreover, we have three cases:

• Case $S = 0$: The Ricci tensor of all the metrics g^λ has two different constant eigenvalues, and in particular the scalar curvature is constant. The Heisenberg Group $G(\mathbb{H})$ is an example for this case.

• Case $S > 0$: The cone metric $r^2 g^1 + dr^2$ on $M \times \mathbb{R}$ is hyper-Kähler, i.e. $\{g^1, K_1, K_2, K_3\}$ is a 3-Sasaki structure.

• Case $S < 0$: The cone metric $r^2 g^{-1} - dr^2$ on $M \times \mathbb{R}$ is hyper-Kähler with signature $(-4, 4n)$. In this case the metrics g^{-1} and $g^{-\frac{2}{4n+6}}$ are Einstein.

In the 3 cases the complementary distribution V is integrable and thus defines a 3-dimensional foliation. The space of leaves is a quaternionic Kähler orbifold.

Existence of an integrable representative in [g] for a general QC str.

• The existence of an integrable g in $[g]$ is equivalent to the existence of a function f on M satisfying a certain system of second order differential equations. Namely

$$T^0(X,Y) \; = \; f^{-1}[\nabla df]_{[sym][-1]}$$

$$U(X,Y) = \; (2f)^{-1}[\nabla df - 2f^{-1}df \otimes df]_{[3][0]}.$$

Here T^0 and U denote the different components of the torsion $T(\xi,.)$.

Thm **[Ivanov, Minchev, Vassilev : work in progress]** **:** If the QC mfd M could be imbedded in a hyper-Kähler mfd then there exists an integrable representative $g \in [g]$.

Note: All the known examples (even the non-explicit one given by LeBrun) are imbedded in certain hyper-Kähler mfds.

Back to the Heisenberg group $G(\mathbb{H})$

- H - the left invariant QC-distribution
- ∇: the left invariant connection
- The sub-Laplacian:

$$\Delta f := tr_H(\nabla df) = \sum_{i=1}^{4n} X_i^2(f),$$

where $\{X_i\}$ is some left invariant ONB of H.

- Δ is not elliptic
- Δ is hypoelliptic [by applying Hörmander's theorem (1967)], i.e. the solutions of the equation $\Delta f = g$ with $g \in C^\infty$ are also C^∞.

• The quaternionic Yamabe equation:

$$\Delta f = -\lambda f^{(2^*-1)}, \qquad \lambda > 0,$$

where $2^* = \frac{2Q}{Q-2}$, $Q = 4n+6$ is the homogeneous dimension.

Thm. Let $\Omega \subset G(\mathbb{H})$ be an open set. There exists a constant $S > 0$ such that for any $f \in C_0^\infty(\Omega)$

$$\left(\int_\Omega |f|^{2^*} \, dv\right)^{1/2^*} \leq S \, \left(\int_\Omega |\nabla f|^2 \, dv\right)^{1/2}$$ [Folland - Stein 1974]

Problems:

- Optimal value for S? (the best constant)
- For which f does the equality hold? (extremals)

• The above problem is equivalent to the problem of describing all representatives $g \in [g]$ of the canonical QC-str. on $G(\mathbb{H})$ having constant nonzero QC-scalar curvature.

Consider S^{4n+3} with its canonical 3-Sasaki structure.

- The Cayley transform $\mathcal{C} : S^{4n+3} \setminus \{\infty\} \to G(\mathbb{H})$ is a QC-isomorphism.

Thm.[Ivanov, Minchev, Vassilev 2007] Up to a constant all representative $g \in [g]$ with constant QC - scalar curvature of the canonical QC str. on S^7 are given by a QC-automorphism of the sphere S^7.

Note. The group of the QC-automorphisms of S^7 is known to be $Sp(2,1)$, hence the above theorem gives an explicit formula for all representatives $g \in [g]$ with constant QC-scalar curvature.

Corollary : On the 7-dimensional Heisenberg group $G(\mathbb{H}) = \mathbb{H} \times Im(\mathbb{H})$, in the L^2 Folland-Stein imbedding theorem

- the best constant is $S = \frac{2\sqrt{3}}{\pi^{3/5}}$

- all extremals are obtained by left translations and dilations on $G(\mathbb{H})$ from the function

$$\frac{2^{11}\sqrt{3}}{\pi^{3/5}} \left[\, (1 + |p|^2)^2 \; + \; |w|^2 \, \right]^{-2}, \qquad (p, w) \in \mathbb{H} \times Im(\mathbb{H}).$$