Tropical Hurwitz numbers

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Lausanne, June 2008

Hurwitz numbers

Definition

Pick $p_1, \dots p_r \in \mathbb{P}^1$. Define $H_d^g(\mu,\nu) :=$ weighted number of degree d covers $C \xrightarrow{\pi} \mathbb{P}^1$ such that

- C is a smooth connected curve of genus g,
- π ramifies with profile μ over p_1,
- π ramifies with profile ν over p_2,
- π has simple ramification over p_i, $3 \leq i \leq r$,
- π is unramified over $\mathbb{P}^1 \setminus \{p_1, \dots, p_r\}$.

Each cover π is weighted by $1/|Aut(\pi)|$.

r, μ, ν and g are related by the Riemann-Hurwitz-formula

$$2 - 2g = 2 - r + |\mu| + |\nu|.$$

$H_d^g(\mu,\nu)$ does not depend on the position of the points p_i.

- M= moduli space of genus g degree d covers of $\mathbb{P}^1$
- branch map $\text{br} : M \to \text{Sym}^r(\mathbb{P}^1)$
- $\deg(\text{br}) =$ Hurwitz number $H_d^g(1^d, 1^d)$
- $M \subset \overline{M}_{g,0}(\mathbb{P}^1, d)$
- br extends to $\overline{M}_{g,0}(\mathbb{P}^1, d)$.

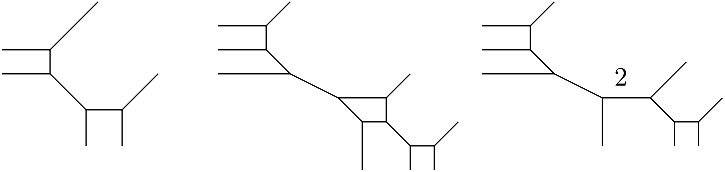
Aim:

- Define tropical Hurwitz numbers $H^g_{d,\text{trop}}(\mu, \nu)$ as $\deg(\text{br}_{\text{trop}})$.
- Show $H^g_{d,\text{ trop}}(\mu, \nu) = H^g_d(\mu, \nu)$.

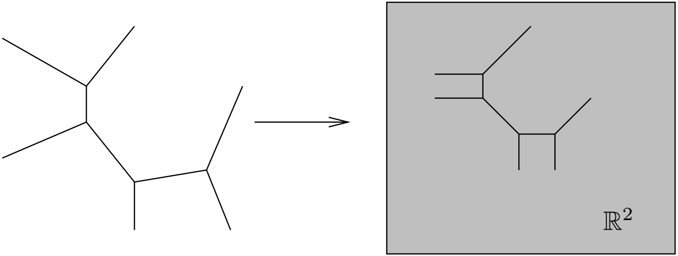
open questions

- Fix $|\mu| = s$ and $|\nu| = t$.
- $N = \{(\mu, \nu) \in \mathbb{N}^{s+t} \mid \mu_1 + \ldots + \mu_s = \nu_1 + \ldots + \nu_t = d\}$
- $N \to \mathbb{Q} : (\mu, \nu) \mapsto H_d^g(\mu, \nu)$ is piece-wise polynomial
- wall-crossing formulas not known for $g > 0$
- lower bounds for degree conjectured
- the new tropical approach seems promising! (work in progress)

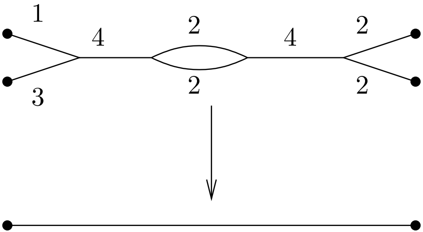
embedded tropical plane curves



parametrized tropical plane curves



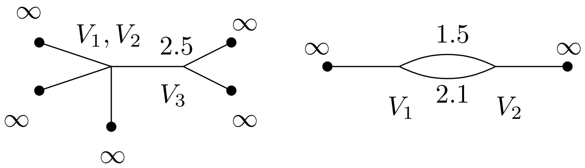
tropical maps to $\mathbb{P}^1$



Definition

An abstract tropical curve with labelled vertices of genus g is

- a connected graph of genus g
- no 2-valent vertices
- edges have length
- ends (edges ending at a 1-valent vertex) have length ∞
- bounded edges have length in $\mathbb{R}_{>0}$
- r-valent vertex ($r \geq 3$) gets $r-2$ labels.



Definition

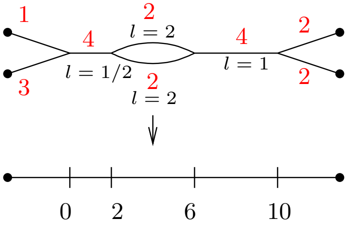
$\Delta \subset \mathbb{Z} \setminus \{0\}$ a multiset satisfying balancing. A parametrized tropical curve of genus g and degree Δ in $\mathbb{P}^1$ with labelled vertices is a tuple (Γ, h) such that

- Γ is an abstract tropical curve of genus g with $\#\Delta$ ends
- $h : \Gamma \to \mathbb{R} \cup \{\pm\infty\}$ such that
  - $h(\Gamma \setminus \{1-\text{valent vertices}\}) \subset \mathbb{R}$
  - h maps edge e of length $l(e)$ affinely to a line segment of length $\omega(e) \cdot l(e)$ where $\omega(e) \in \mathbb{N}$ is called weight
  - weights on both sides of a vertex are equal
  - multiset of directions of ends equals Δ.

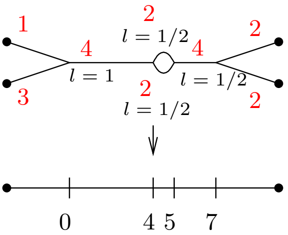
The space of all such is called $M_{g,0,\text{trop}}(\mathbb{P}^1, \Delta)$.

$$\begin{aligned} \text{br}_{\text{trop}} : M_{g,0,\text{trop}}(\mathbb{P}^1, \Delta) &\to \mathbb{R}^{\#\Delta+2g-2} \\ (\Gamma, h) &\mapsto (h(V_1), \ldots, h(V_{\#\Delta+2g-2})) \end{aligned}$$

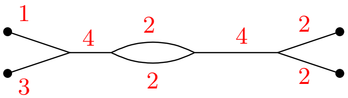
Now fix points (e.g. (0,2,6,10)) and count preimages under br.



Fix different points (e.g. (0,4,5,7)):



Conclusion: we only need to count weighted graphs:

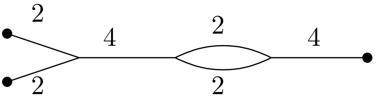


Each such graph comes with a multiplicity:

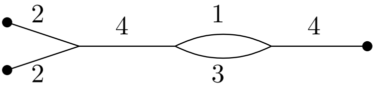
$$\prod \text{weights of bounded edges} \cdot \text{automorphism factors}$$

$$4 \cdot 2 \cdot 2 \cdot 4 \cdot \frac{1}{2} \cdot \frac{1}{2}$$

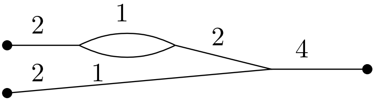
$H^1_{4,\,\mathrm{trop}}((2,2),4) = 14$



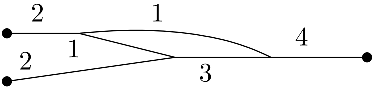
$$4 \cdot 2 \cdot 2 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = 4$$



$$4 \cdot 3 \cdot \tfrac{1}{2} = 6$$



$$1 \cdot 1 \cdot 2 \cdot \tfrac{1}{2} = 1$$



$$1 \cdot 1 \cdot 3 = 3$$

References:

- Torsten Ekedahl, Sergei Lando, Michael Shapiro, and Alek Vainshtein. *Hurwitz numbers and intersections on moduli spaces of curves.* Invent. Math., 146:297-327, 2001.
- Grigory Mikhalkin. *Tropical geometry and its applications.* In M. Sanz-Sole et al., editor, Invited lectures v. II, Proceedings of the ICM Madrid, 827-852, 2006.
- Renzo Cavalieri, Paul Johnson and Hannah Markwig. *Tropical Hurwitz numbers.* Preprint, arXiv:0804.0579, 2008.

THANK YOU

Tropical one-day workshop in Göttingen: July 9, 2008.
http://crcg.de/workshops/tropical/