Orbifold String Topology and Hochschild cohomology

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Contents:

- Recall the definition of String topology on manifolds
  - Recall its relation with Hochschild cohomology and
  - Symplectic field theory
- Orbifold String Topology
  - Topological definition
  - Relation with Hochschild cohomology

String Topology

Let M^d be a differentiable, compact and oriented manifold.

$$\mathcal{L}M := \text{piece-wise smooth maps} \;\; S^1 \to M$$

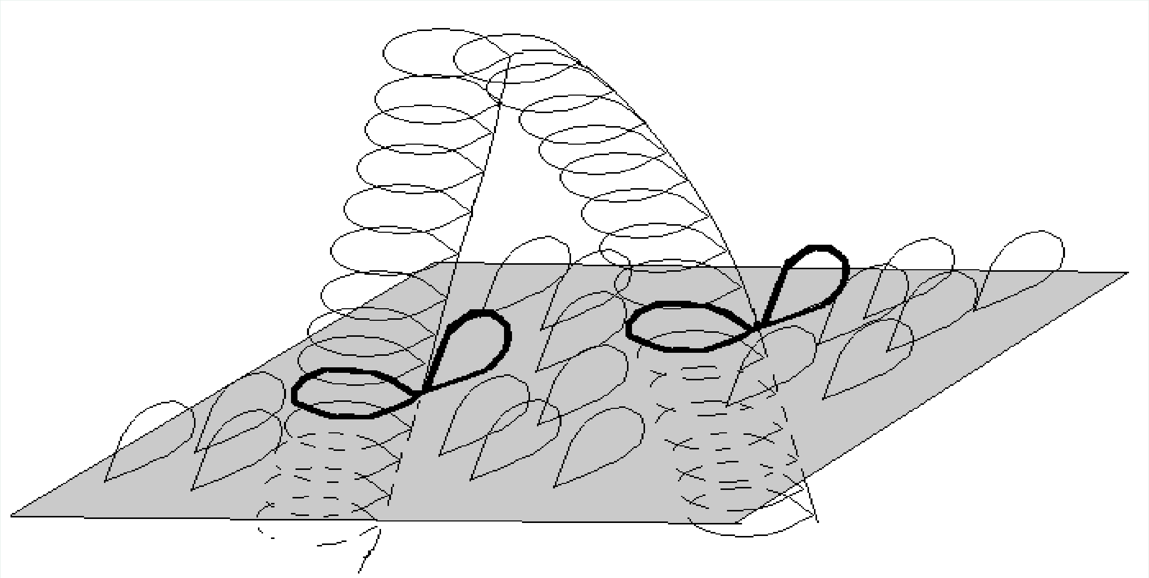
$$\mathbb{H}_*(\mathcal{L}M) := H_{*+d}(\mathcal{L}M, \mathbb{Z})$$

Chas and Sullivan (99) defined a ring structure

$$\mathbb{H}_p(\mathcal{L}M) \times \mathbb{H}_q(\mathcal{L}M) \xrightarrow{\bullet} \mathbb{H}_{p+q}(\mathcal{L}M)$$

that they called "loop product".

Chas and Sullivan defined the loop product at the level of chains by performing the Pontryagin product at the points of transversal intersection of the evaluation at zero of two cylces in $\mathcal{L}M$; pictorially



Considering also the degree one map

$$\begin{aligned} \Delta : \mathbb{H}_*(\mathcal{L}M) &\rightarrow \mathbb{H}_{*+1}(\mathcal{L}M) \\ \alpha &\mapsto \tau_*(\Theta \otimes \alpha) \end{aligned}$$

where

$$\begin{aligned} \tau : S^1 \times \mathcal{L}M &\rightarrow \mathcal{L}M \\ (t, f) &\mapsto f(t + \cdot) \end{aligned}$$

and Θ a generator of $H_1(S^1, \mathbb{Z})$.

Theorem (Chas-Sullivan 99). *The loop product $\bullet$ and the operator Δ makes $\mathbb{H}_*(\mathcal{L}M)$ into a Batalin-Vilkovisky algebra, namely*

- *the loop product $\bullet$ is graded commutative associative*
- $\Delta \circ \Delta = 0$
- $\{a, b\} := (-1)^{|a|}\Delta(a \bullet b) + (-1)^{|a|}\Delta(a) \bullet b - a \bullet \Delta(b)$ *makes $\mathbb{H}_*(\mathcal{L}M)$ into a Gerstenhaber algebra*

If one considers the fibration $\Omega M \to \mathcal{L}M \to M$ and the ring structures

$$(H_*(M), \wedge) \quad \text{intersection of cycles}$$

$$(H_*(\Omega M), \times) \quad \text{Pontryagin product}$$

one has the following maps

$$(H_*(M), \wedge) \xrightarrow{\text{constant loops}} (\mathbb{H}_*(\mathcal{L}M), \bullet) \xrightarrow{\text{inter. fiber}} (H_*(\Omega M), \times)$$

Example (Cohen-Jones-Yan 02)**.**

$$\mathbb{H}(\mathcal{L}S^{2n}) \cong \Lambda[b] \otimes \mathbb{Z}[a, v]/(a^2, ab, 2av) \qquad \mathbb{H}(\mathcal{L}S^{2n+1}) \cong \Lambda[a] \otimes \mathbb{Z}[c]$$

Hochschild cohomology

Let A be a DG-algebra and M an A-bimodule. Consider the complex

$$C_n(A, M) := M \otimes_{\mathbb{Z}} A^{\otimes n} \text{ and differential}$$

$$\begin{aligned} m \otimes a_1 \otimes \cdots \otimes a_n \overset{\delta}{\mapsto} \; & ma_1 \otimes a_2 \otimes \cdots \otimes a_n + \\ & \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ & + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

Then the Hochschild homology of the pair (A, M) is

$$HH_*(A, M) =: H_*(\mathrm{Tot}(C_*(A, M), \delta))$$

Dualizing, define

$$C^*(A, M) := \mathrm{Hom}_{\mathbb{Z}}(A^{\otimes n}, M) \text{ and differential } f \mapsto \overline{\delta} f$$

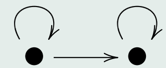
$$\begin{aligned} \overline{\delta} f(a_1 \otimes \cdots \otimes a_n) \; := \; & f(a_2 \otimes \cdots \otimes a_n)a_1 + \\ & \sum_{i=1}^{n-1} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ & +(-1)^n a_n f(a_1 \otimes \cdots \otimes a_{n-1}). \end{aligned}$$

Then the Hochschild cohomology of the pair (A, M) is

$$HH^*(A, M) =: H_*(\mathrm{Tot}(C^*(A, M), \overline{\delta}))$$

Note that in the case that $M = A$ the Hochschild cohomology

$HH^*(A, A)$ is a graded algebra.

Jones [87] considered the category  together with the simplicial set λ_* associated to it and showed that

$$\mathrm{Maps}([0,1], M) \simeq ||\mathrm{Maps}(\lambda_*, M)||$$

where the right hand side is the geometrical realization of the cosimplicial space $\mathrm{Maps}(\lambda_*, M)$.

Restricting to the maps that start and finish at the same point, Jones obtained a cosimplicial space $\mathbb{X}^\bullet$ with $\mathbb{X}^n \cong M^{n+1}$ and such that

$$\mathcal{L}M \simeq ||\mathbb{X}^\bullet||,$$

applying the cochains functor, Jones further proved

Theorem (Jones 87)**.** *If M is simply connected, then*

$$H^*(\mathcal{L}M) \cong H_*(\mathrm{Tot}\ C^*(\mathbb{X}^\bullet)) \cong HH_*(C^*M, C^*M)$$

as graded abelian groups.

Dualizing, Cohen and Jones [02] showed

$$H_*(\mathcal{L}M) \cong HH^*(C^*M, C_*M)$$

and endowing C_*M with a product structure via transversal intersection $\pitchfork$, they showed

Theorem (Cohen-Jones 02)**.** *If M is simply connected, then there is an isomorphism of graded algebras*

$$\mathbb{H}_*(\mathcal{L}M) \cong HH^*(C^*M, C_*M)$$

Moreover, Cohen and Jones also proved that the loop product can be realized at the level of topological spaces as the composition

$$\mathcal{L}M \times \mathcal{L}M \xrightarrow{\overline{\Delta}_!} (\mathcal{L}M \times_M \mathcal{L}M)^{TM} \xrightarrow{\circ} \mathcal{L}M^{TM}$$

where $\overline{\Delta}_!$ is the umkehr map of the inclusion

$$\begin{array}{ccc} \mathcal{L}M \times_M \mathcal{L}M & \xrightarrow{\overline{\Delta}} & \mathcal{L}M \times \mathcal{L}M \\ \downarrow e_\infty & & \downarrow e \\ M & \xrightarrow[\Delta]{} & M \times M, \end{array}$$

$(\mathcal{L}M \times_M \mathcal{L}M)^{TM}$ and $\mathcal{L}M^{TM}$ are the Thom-Pontryagin spaces associated to the bundles $e_\infty^* TM$ and $e^* TM$ respectively, and $\circ$ is the composition of loops.

In homology one gets

$$H_*(\mathcal{L}M \times \mathcal{L}M) \to H_*\left((\mathcal{L}M \times_M \mathcal{L}M)^{TM}\right) \to H_*(\mathcal{L}M^{TM}) \cong H_{*-d}(\mathcal{L}M)$$

By taking the Thom-spectra $\mathcal{L}M^{-TM}$ associated to the virtual bundle $-TM$, then one has

$$\mathcal{L}M^{-TM} \wedge \mathcal{L}M^{-TM} \to \mathcal{L}M^{TM-\Delta^*(TM+TM)} \simeq \mathcal{L}M^{-TM}$$

making the Thom-spectra $\mathcal{L}M^{-TM}$ into a ring spectra.

Then the homology $H_*(\mathcal{L}M^{-TM})$ becomes an algebra and we have that

Theorem (Cohen-Jones 02). *If M is simply connected, then there is an isomorphism of graded algebras*

$$HH^*(C^*M, C^*M) \cong H_*(\mathcal{L}M^{-TM})$$

Note:
In the case that $A = C^*M$ is the dga of integer valued singular cochains on a manifold M, we show that

$$HH^*(C^*M, C^*M) \cong \mathrm{Ext}_{C^*M^e}(C^*M, C^*M)$$

where $C^*M^e = C^*M \otimes_{\mathbb{Z}} C^*M^{op}$ and C^*M is a C^*M^e module in the natural way.

The problem is that C^*M is not a free $\mathbb{Z}$-module...
But as C^*M is the dual of a free $\mathbb{Z}$-module, we could construct a spectral sequence that showed that the Hochschild cohomology of the cochains could be understood as an Ext ring.

Relation with Symplectic Field Theory

Let (P, ω) be a symplectic manifold and

$$H : \mathbb{R}/\mathbb{Z} \times P \to \mathbb{R}$$

a time dependent and 1-periodic smooth hamiltonian. Let X_H be the vector field defined by

$$\omega(X_H, \cdot) = -dH$$

and let the 1-periodic solutions be

$$P(H) = \{x \in \mathcal{L}P \mid \dot{x}(t) = X_H(t, x(t)) \; \forall t \in \mathbb{R}\}.$$

(Roughly speaking) Floer defined a chain complex generated by the elements in $P(H)$ and used a Cauchy-Riemann type of PDE for cylindrical maps

$$U : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \to P$$

connecting $x, y \in P(H)$, in order to select and count particular trajectories in the free loop space and thus defining a cain boundary operator.

The homology of this complex $FH_*(P, \omega)$ is known as the Floer homology of (P, ω)

Considering the case $(P,\omega) = (T^*M, \omega_{can})$, then

Theorem (Viterbo 96, Salamon-Weber 03, Abbondandolo-Scwhartz 04)**.** *There is a isomorphism*

$$H_*(\mathcal{L}M) \cong HF_*(T^*M, \omega_{can})$$

as graded abelian groups.

Adding the pair of pants product structure

Theorem (Abbondandolo-Scwhartz 05)**.** *There is an isomorphism*

$$\mathbb{H}_*(\mathcal{L}M) \cong HF_*(T^*M, \omega_{can})$$

as graded algebras.

Orbifold String Topology

Let us consider the orbifold $[M/G]$ where G is a finite group acting smoothly on M. Let the loop orbifold be

$$\mathcal{L}[M/G] := \mathrm{Maps}([\mathbb{R}/\mathbb{Z}], [M/G]) = [P_G M/G]$$

with

$$P_G M = \bigsqcup_{g \in G} P_g M \times \{g\} \quad \text{and} \quad P_g M := \{f : \mathbb{R} \to M \mid f(t)g = f(t+1)\}$$

together with the G-action

$$P_G M \times G \to P_G M \qquad ((f,g),h) \mapsto (f \cdot h, h^{-1}gh).$$

Proposition (Lupercio-U-Xicoténcatl 05)**.** *There is a homotopy equivalence*

$$\mathcal{L}(M \times_G EG) \simeq P_G M \times_G EG.$$

Consider the diagram

$$\begin{array}{ccccc} P_gM \times P_hM & \xleftarrow{\overline{\Delta}} & P_gM \times_M P_hM & \xrightarrow{\text{comp}} & P_{gh}M \\ \downarrow{\scriptstyle e_1\times e_0} & & \downarrow & & \downarrow \\ M \times M & \xleftarrow{\Delta} & M & \xrightarrow{=} & M \end{array}$$

with the left part is a pullback square and the map on the right is the concatenation of paths.

If we denote $H^G_*(\cdot) := H_*(\cdot \times_E EG)$, then we can consider the loop product that define the composition of the following homomorphisms:

$$H^{G\times G}_*(P_GM \times P_GM) \xrightarrow{\text{transfer}} H^G_*(P_GM \times P_GM) \xrightarrow{\overline{\Delta}_!} H^G_*\left((P_GM \times_M P_GM)^{TM}\right)$$

$$\xrightarrow{\text{comp}} \quad H^G_*(P_GM^{TM}) \longrightarrow H^G_{*-d}(P_GM)$$

where the map "transfer" is the one induced by the diagonal homomorphism $G \to G \times G$.

The previous loop product, together with the natural action of S^1, gives us

Theorem (Lupercio-U-Xicoténcatl 05). *The homology of the loop orbifold*

$$H_*(\mathcal{L}[M/G], \mathbb{Z})$$

can be endowed with the structure of a BV-algebra.

This theorem was generalized for general oriented differentiable stacks

Theorem (Behrend-Ginot-Noohi-Xu 07). *The homology of the loop stack*

$$H_*(\mathcal{L}\mathcal{X}, \mathbb{Z})$$

of a differentiable oriented stack $\mathcal{X}$ can be endowed with the structure of a BV-algebra.

When $M =$ point and the group G is abelian we have that

$$\mathcal{L}[*/G] = [G^{\mathrm{ad}}/G] \cong G \times [*/G] \qquad\qquad \mathcal{L}BG \simeq G \times BG.$$

Therefore the algebra structure on the orbifold loops is given by

$$H_*(\mathcal{L}[*/G]) \cong \mathbb{Z}G \otimes_{\mathbb{Z}} H_*(G, \mathbb{Z})$$

with the algebraic structure on the homology of G defined by the transfer map of the diagonal homomorphism $G \to G \times G$.

Note that in the case $G = \mathbb{Z}/n$ the algebraic structure defined by the transfer map on $H_*(\mathbb{Z}/n, \mathbb{Z})$ is trivial !!!!

Obifold String Topology REVISITED

Let us consider the singular cochains of the orbifold $[M/G]$ as the smash product DG-algebra $C^*M\#G$, with

$$C^*M\#G := C^*M \times G \quad \text{as a set, an with product}$$

$$(\alpha, g) \cdot (\beta, h) := (\alpha \cup g^*\beta, gh).$$

Proposal [Ángel-Backelin-U] The algebraic structure of the string topology on the orbifold $[M/G]$ should be recovered from the algebra

$$HH^*(C^*M\#G, C^*M\#G).$$

Topological counterpart

Take $EG_1 \subset \cdots \subset EG_n \subset EG_{n+1} \subset \cdots EG$ an approximation to the universal principal bundle EG by finite dimensional manifolds.

Consider the diagram

$$\begin{array}{ccccc}
(P_GM \times_G EG_n) \times (P_GM \times_G EG_n) & \xleftarrow{\overline{\Delta}} & (P_GM \times P_GM) \times_G EG_n & \longrightarrow & P_GM \times_G E \\
\downarrow {\scriptstyle e_1 \times e_0} & & \downarrow & & \downarrow \\
(M \times_G EG_n) \times (M \times_G EG_n) & \xleftarrow{\Delta} & M \times_G EG_n & \longrightarrow & M \times_G E
\end{array}$$

that defines a map

$$(P_GM \times_G EG_n) \times (P_GM \times_G EG_n) \longrightarrow P_GM \times_G EG_n{}^{T(M \times_G EG_n)}$$

which defines a ring spectrum

$$P_GM \times_G EG_n{}^{-T(M \times_G EG_n)} \wedge P_GM \times_G EG_n{}^{-T(M \times_G EG_n)} \longrightarrow P_GM \times_G EG_n{}^{-T(M \times_G EG_n)}.$$

Assemble the spectra ${P_G M \times_G EG_n}^{-T(M\times_G EG_n)}$ into an inverse system of ring spectra

$$\cdots \longleftarrow {P_G M \times_G EG_n}^{-T(M\times_G EG_n)} \longleftarrow {P_G M \times_G EG_{n+1}}^{-T(M\times_G EG_{n+1})} \longleftarrow \cdots$$

and denote this inverse system (ring pro-spectra) by

$$\mathcal{L}(M\times_G EG)^{-T(M\times_G EG)} = P_G M \times_G EG^{-T(M\times_G EG)} := \lim_{\leftarrow n} {P_G M \times_G EG_n}^{-T(M\times_G EG_n)}.$$

Theorem (Ángel-Backelin-U)**.** *For M simply connected, there is an isomorphism of graded algebras*

$$H_*^{\text{pro}}\left(\mathcal{L}(M \times_G EG)^{-T(M\times_G EG)}\right) \cong HH^*(C^*M\#G, C^*M\#G).$$

Corollary. *When the coefficients are in* $\mathbb{Q}$*, there is an isomorphism of algebras*

$$H_*(\mathcal{L}[M/G], \mathbb{Q}) \cong H_*^{\mathrm{pro}}\left(\mathcal{L}(M \times_G EG)^{-T(M \times_G EG)}, \mathbb{Q}\right)$$

where the algebra in the left-hand side is the one defined by Lupercio-U-Xicoténcatl and Behrend-Ginot-Noohi-Xu.

If N is a manifold with finite fundamental group $G = \pi_1 N$, and $\widetilde{N}$ is its universal cover, then we have:

Corollary.

$$H_*(\mathcal{L}N^{-TN}, \mathbb{Z}) \cong HH^*(C^*\widetilde{N}\#G, C^*\widetilde{N}\#G)$$

where on the left hand side we have the String topology of N *defined topologically, and on the right hand side we have the Hochschild cohomology of the dg-ring* $C^*\widetilde{N}\#G$*.*

This is the version in Hochschild cohomology of String Topology when the manifold is not simply connected.

Proposition (Ángel-Backelin-U). *There is an isomorphism of algebras*

$$H_*^{\mathrm{pro}}\left(\mathcal{L}BG^{-TBG}, \mathbb{Z}\right) \cong \bigoplus_{(g)} H^*(BC(g), \mathbb{Z}).$$

where $C(g) = \{h \in G \mid gh = hg\}$, (g) *runs over the conjugacy classes of elements in* G *and the algebra structure of the right-hand side is given by the pull-push construction, i.e.*

For $\alpha \in H^*(BC(g), \mathbb{Z})$ and $\beta \in H^*(BC(h), \mathbb{Z})$, then

$$(\alpha \cdot \beta)_{(gh)} := \frac{|C(gh)|}{|C(g) \cap C(h)|} \mathrm{cor}_{C(g)\cap C(h)}^{C(gh)} \left(\mathrm{res}_{C(g)\cap C(h)}^{C(g)} \alpha \cup \mathrm{res}_{C(g)\cap C(h)}^{C(h)} \beta\right).$$

Note that if G is abelian then there is an isomorphism of algebras

$$H_*^{\mathrm{pro}}\left(\mathcal{L}BG^{-TBG}\right) \cong \mathbb{Z}G \otimes_{\mathbb{Z}} H^*(BG, \mathbb{Z}).$$

End Remarks

- Morita equivalent groupoids may not induce Morita equivalent dg-rings. For example, let $G = \mathbb{Z}/2$ acts on $M = S^1$ by the antipodal action; but $C^*(M/G)$ is not Morita equivalent to $C^*M\#G$.
- We believe that our results could be generalized of compact manifolds with non-finite fundamental group, but we have not resolved some issues about dualization on the cochains of the universal cover.
- For orbifolds defined through a Proper Lie Foliation groupoids $\mathcal{G}$, one can construct a candidate for the cochains $C^*\mathcal{G}$ that behaves well under the Eilenberg-Moore spectral sequence. But there is no proof yet that the string topology of $\mathcal{G}$ would be isomorphic to the Hochschild cohomology of $C^*\mathcal{G}$.

END

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