The Structure of Fusion Categories via Topological Quantum Field Theories

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D. Kan

Definition

An adjunction is a pair of functors

$$F : C \leftrightarrows D : G$$

and a natural bijection

$$Hom_D(Fx, y) \cong Hom_C(x, Gy).$$

F is left adjoint to G.

Equivalent Formulation

$$Hom_D(Fx, Fx) \cong Hom_C(x, GFx) \qquad Hom_C(Gy, Gy) \cong Hom_D(FGy, y)$$

$$id_{Fx} \mapsto (\eta_x : x \to GFx) \qquad id_{Gy} \mapsto (\varepsilon_y : FGy \to y)$$

Equivalent Formulation

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Natural transformations...

unit $\eta : id_C \to GF$ counit $\varepsilon : FG \to id_D$

Satisfying equations...

$$F \stackrel{1*\eta}{\to} FGF \stackrel{\varepsilon*1}{\to} F = F \stackrel{id}{\to} F$$
$$G \stackrel{\eta*1}{\to} GFG \stackrel{1*\varepsilon}{\to} G = G \stackrel{id}{\to} G$$

Duality in any bicategory

Definition

An adjunction is a pair of 1-morphisms

$$F : C \leftrightarrows D : G$$

and 2-morphisms

$$\eta : id_C \to GF \quad \varepsilon : FG \to id_D$$

satisfying 'Zig-Zag' equations:

$$F \stackrel{1*\eta}{\to} FGF \stackrel{\varepsilon*1}{\to} F = F \stackrel{id}{\to} F$$

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Higher Category Theory

Use the theory of (∞, n)-categories.

Generalizes both topological spaces and categories.

Hueristically:

C. Barwick

C. Rezk

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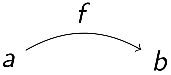

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- 2-morphisms, 3-morphisms, etc.
- compositions...

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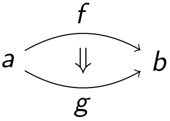
(invertible above n)



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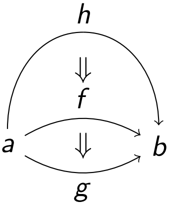

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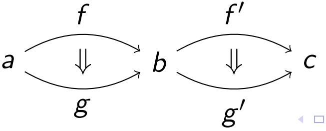

Exmaples of (∞, n)-categories

Example

Cat the 2-category of small categories.
More generally, any bicategory.

Example (Spaces = $(\infty, 0)$-categories)

X a space

- objects = points of X
- 1-morphisms = paths in X
- 2-morphisms = paths between paths
- etc.

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Monoidal Category $(M, \otimes) \rightsquigarrow$ one object bicategory BM.

- 1-morphisms $=$ objects of M
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Dual objects in $M \leftrightarrow$ dual 1-morphisms in BM: x, x^*, and...

$$\text{coevaluation} \quad \eta : 1 \to x \otimes x^* \qquad \text{evaluation} \quad \varepsilon : x^* \otimes x \to 1$$

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Example

$M = Vect$, a vector space x is *dualizable* $\Leftrightarrow$ x is finite dimensional

Fusion Categories

Definition

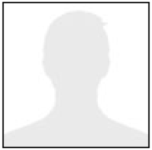
A Fusion Category is a monoidal semi-simple k-linear category, with

- finitely many isom. classes of simples,
- $End(1) \cong k$,
- left and right duals for all objects.

For simplicity, $k = \mathbb{C}$.
Sources of Fusion Categories:

- Quantum Groups
- Operator Algebras
- Conformal Field Theory
- Representations of Loop Groups
- Conformal Nets

P. Etingof



D. Nikshych

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Theorem (Etingof-Nikshych-Ostrik)

In any Fusion category, the functor

$$X \mapsto X^{****}$$

is canonically monoidally equivalent to id.

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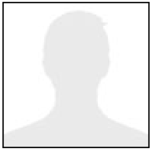

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Why?

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A fusion category is pivotal if it admits a pivotal structure, i.e. a natural monoidal isomorphism $X \cong X^{**}$.

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Conjecture (ENO)

All fusion categories are pivotal.

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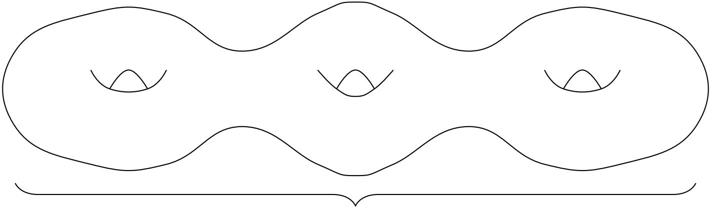
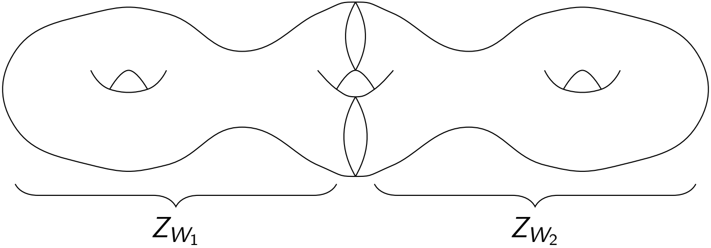
Still Open

Manifold Invariants



$$Z_W$$

Manifold Invariants



Locality of manifold invariants:

Reconstruct Z_W from Z_{W_1} and Z_{W_2}?

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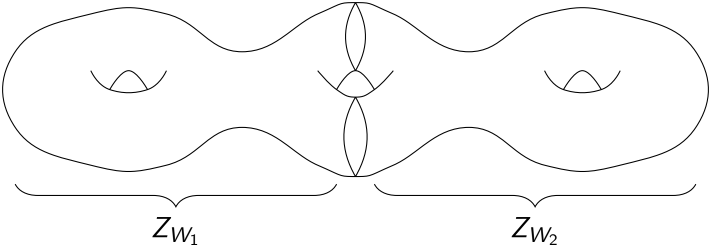

Locality of manifold invariants:

Reconstruct Z_W from Z_{W_1} and Z_{W_2}?

$$Z_W = \langle Z_{W_1}, Z_{W_2} \rangle$$

The Cobordism Category

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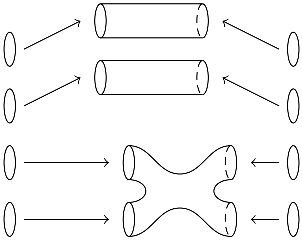

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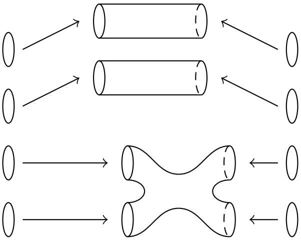


variants:

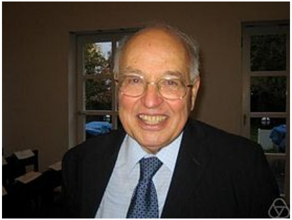
- extra structures: orientations, spin structures, etc
- higher categories of cobordisms

Topological Quantum Field Theories

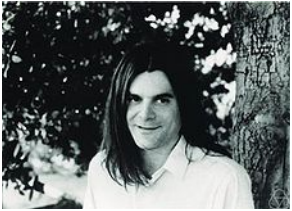
Definition

A TQFT is a symmetric monoidal functor:

$$\underbrace{\text{Bord}}_{\text{Cobordism Category}} \to \underbrace{\text{C}}_{\text{Target Category}}$$



M. Atiyah



G. Segal

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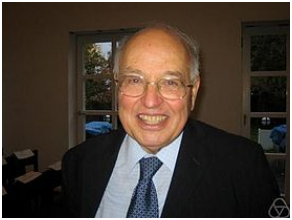
$$\emptyset \in \mathrm{Bord} \mapsto 1 \in \mathrm{C}$$

$$M \text{ closed} \mapsto (1 \stackrel{Z_M}{\to} 1)$$



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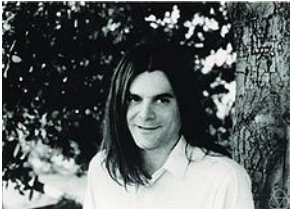

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Distinguishing Manifolds?

- 0D, 1D, and 2D TFTs distinguish manifolds.
- 4D (unitary) TFTs cannot detect smooth structures. [Freedman-Kitaev-Nayak-Slingerland-Walker-Wang]
- 5D (unitary) TFTs can detect, if $\pi_1 = 0$. [Kreck-Teichner]
- $\geq$ 6D (unitary) TFTs cannot detect homotopy type. [Kreck-Teichner]

Open Problem: Can 3D TFTs distinguish 3-manifolds?
Evidence suggest "yes?". [Calegari-Freedman-Walker]

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Turaev-Viro-Barrett-Westbury Construction: a 3D TQFT

Input:
C a Spherical Category

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- Label using data from C
- Weighted average over all labelings gives invariant.

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In 2010...

Theorem (Turaev-Virelizier, Balsam-Kirillov)

This gives a tqft which is local down to 1-manifolds.

Theorem (Douglas-SP-Snyder)

Fusion, Pivotal, and Spherical Categories all give rise to fully local extended 3D TQFTs.

Moreover the *structure* of the TQFTs reflects the structure of fusion categories.

Tangential Structures on Manifolds

a manifold M has a tangent bundle τ
classified by a map

$$M \xrightarrow[\tau]{} BO(n)$$

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$G \to O(n)$

$$\begin{array}{ccc} & & BG \\ & \nearrow & \downarrow \\ M & \xrightarrow[\tau]{} & BO(n) \end{array}$$

- $G = SO(n) \rightsquigarrow$ Orientation
- $G = Spin(n)$ (universal cover of $SO(n)$) $\rightsquigarrow$ Spin structure
- $G = 1 \rightsquigarrow$ framing
- etc

different sorts of fusion categories give different tqfts.

Theorem (Douglas-SP-Snyder)

G	*name of structure*	*kind of category*
$SO(3)^{\dagger}$	*Orientation*	*Spherical*
$SO(2)$	*Combing*	*Pivotal*
$1 = SO(1)$	*Framing*	*Fusion*

† This group might change slightly.

2D (non-local) TQFTs

Theorem (Folklore)

The category of (non-local) oriented 2D tqfts in C is equivalent to category of commutative Frobenius algebras in C.

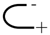
[R. Dijkgraaf, L. Abrams, S. Sawin, B. Dubrovin, Moore-Segal, . . .]

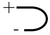
unit multiplication comultiplication counit

1D TQFTs

Theorem (1D Cobordism Hypothesis)

The category of 1D oriented tqfts in C is equivalent to the groupoid of dualizable objects of C, denoted $k(C^{fd})$

coevaluation 

evaluation 

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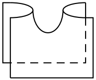
2D Local TQFTs

Like 1D tqfts, but with 2D bordisms too.

- Objects (0-manifolds) have duals



2D Local TQFTs

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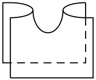
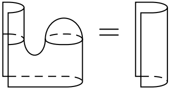
- Objects (0-manifolds) have duals
- 1-morphisms (1-manifolds) also have duals



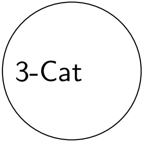
$$F \circ G \overset{}{\Downarrow} \varepsilon \; id$$

$$id \Downarrow \eta \; G \circ F$$

Zig-Zag Equation:



Layers of dualizability



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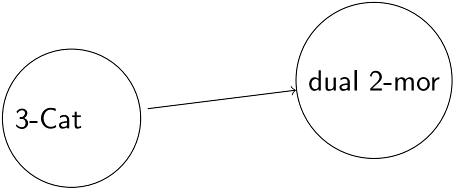

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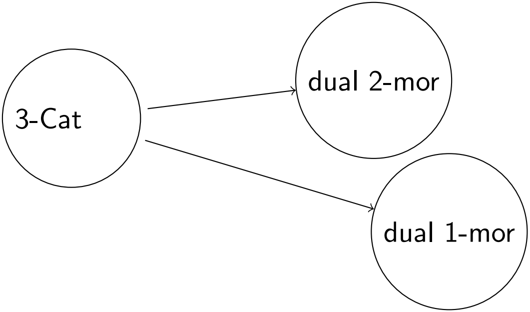

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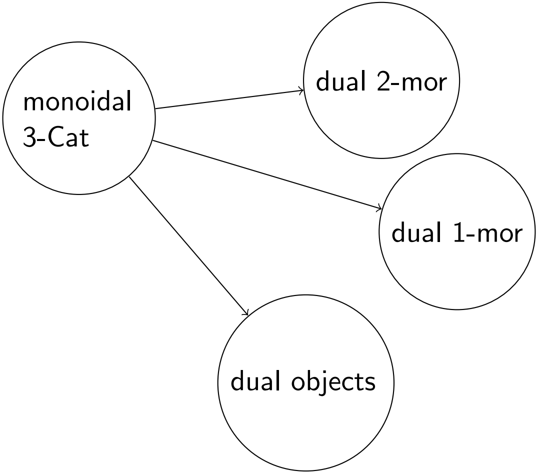


Fully-dualizable

Fully-dualizable is dualizable on all levels:

Definition

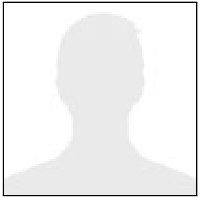
If C is a symmetric monoidal n-category, there is a filtration

$$C^{fd} = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_{n-1} \subseteq C$$

where C_i = the maximal sub-n-category where j-morphisms have both duals if $i \leq j \leq n-1$.

Baez-Dolan Cobordism Hypothesis

J. Baez



J. Dolan

“Bord_n is the free symmetric monoidal n-category with duality”

Theorem (Hopkins-Lurie)

$Fun(\text{Bord}_n^{fr}, \text{C}) \simeq k(\text{C}^{fd})$

M. Hopkins

J. Lurie

Theorem (Douglas-SP-Snyder)

Fusion categories are fully-dualizable objects in the symmetric monoidal 3-category TC. (Tensor Categories)

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Corollary

Fusion categories give rise to fully-local extended 3D tqfts.

What is TC?

The 3-category of Tensor Categories

Example

Algebras, Bimodules, Bimodule maps = a (monoidal) 2-category

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Definition

$TC =$

- objects: Tensor Categories (monoidal k-linear)
- 1-morphisms: Bimodule Categories
- 2-morphisms and 3-morphisms: Bimodule Functors and Bimodule Natural Transformations

Monoidal for *Deligne tensor product*.

Proposition (Douglas-SP-Snyder)

TC is a symmetric monoidal $(\infty, 3)$-category.

A Basic Principle

If G acts on B, then G acts on $Map(B, C)$.

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$O(3)$ acts on Bord_3^{fr} by change of framing.

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A Basic Principle and a Theorem

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$O(3)$ acts on Bord_3^{fr} by change of framing.

$$O(3) \to Aut(k(\mathrm{C}^{fd}))$$

Theorem (Hopkins-Lurie)

$Fun(\mathrm{Bord}_n^G, \mathrm{C}) \simeq [k(\mathrm{C}^{fd})]^{hG}$.

So $O(3)$ acts on the "space" of fusion categories.
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- points in $O(3) \rightsquigarrow$ self-equivalences $k(\mathsf{C}^{fd}) \to k(\mathsf{C}^{fd})$

So $O(3)$ acts on the "space" of fusion categories.
What is the action?

- points in $O(3) \rightsquigarrow$ self-equivalences $k(\mathrm{C}^{fd}) \to k(\mathrm{C}^{fd})$
- paths in $O(3) \rightsquigarrow$ natural isomorphisms
- paths between paths in $O(3) \rightsquigarrow$ natural 2-isomorphism
- etc

In more detail...

- $\pi_0 O(3) = \mathbb{Z}/2$, non-trivial element: $(F, \otimes) \mapsto (F, \otimes^{op})$.
- $\pi_1 O(3)$ gives the *Serre automorphism* (natural automorphism of identity functor)

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  in components
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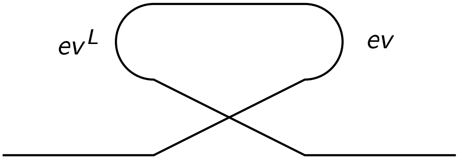
  is an invertible F-F-bimodule category.
- $\pi_2 O(3) = 0$
- $\pi_3 O(3) = \mathbb{Z}$ gives the *anomaly*. $\rightsquigarrow a_F \in \mathbb{C}^\times$

No other data since TC is just a 3-category.



Theorem (Douglas-SP-Snyder)

The Serre Automorphism of a fusion category F is the bimodulification *of*

$$(F, \otimes) \to (F, \otimes)$$
$$x \mapsto x^{**}$$

$\pi_1 O(3) \cong \mathbb{Z}/2 \Rightarrow$ square of the Serre is trivial!

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Corollary

The bimodulification *of* $x \mapsto x^{****}$ *is trivial.*

$\Rightarrow x^{****} \cong D \otimes x \otimes D^{-1}$ for some $\otimes$-invertible object D.

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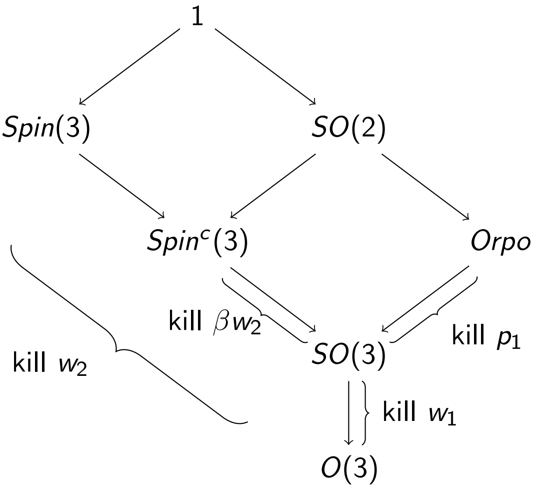
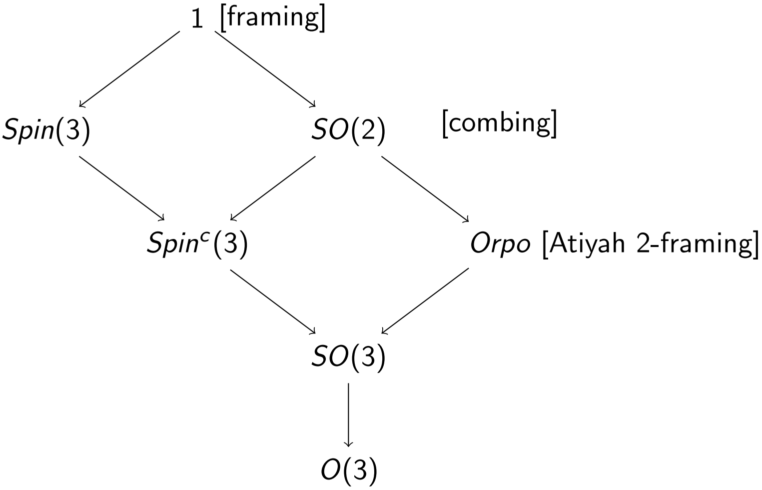
Proposition (Douglas-SP-Snyder)

For fusion categories $D \cong 1$*, so* $x^{****} \cong x$*.*

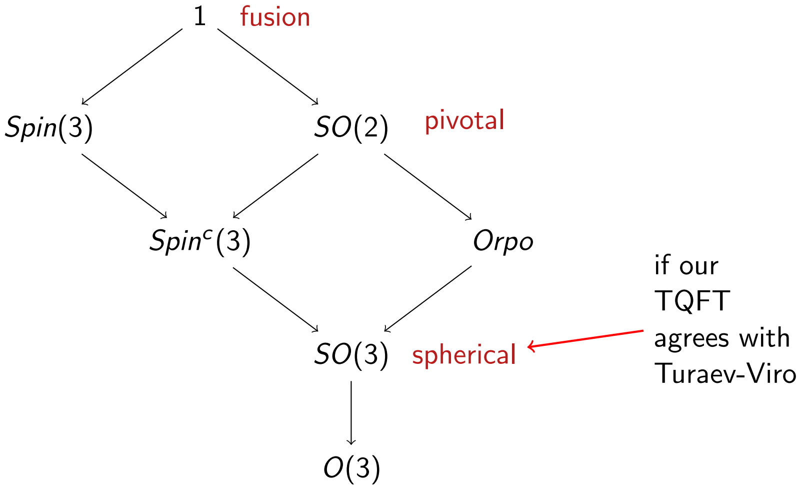
Some 3D structure groups



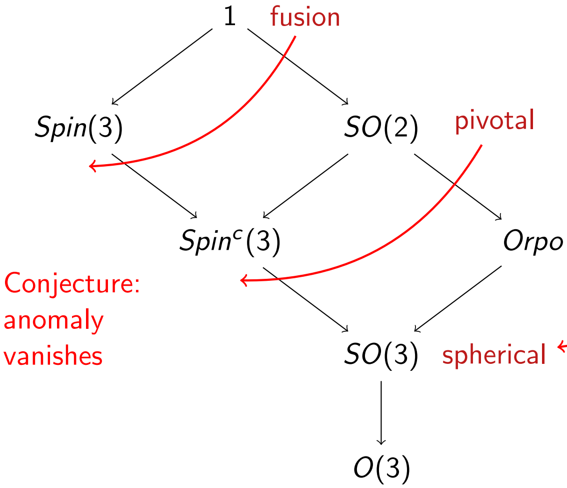
Some 3D structure groups



Some 3D structure groups



Some 3D structure groups



if our TQFT agrees with Turaev-Viro

A new version of ENO conjecture

Conjecture

All framed extended 3D tqfts in TC can be extended to oriented tqfts.

Evidence one dimension lower...

Theorem

All framed extended 2D tqfts in Alg can be extended to oriented tqfts.

The End