Differential graded manifolds, infinity-stacks and generalized geometries

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(notes taken by Christoph Wockel)

June 5, 2009

I dg manifolds - general theory

dg-manifolds are supermanifolds with certain additional structures. First some generalities on supermanifolds and super vectorspaces:

I.1 Linear Algebra

V monoidal category, e.g. Set, Vect, $\mathcal{C}$ a V-enriched category. For objects a, b of $\mathcal{C}$ (write $c, d \in \mathcal{C}$), write$\mathcal{C}(a, b)$ for the hom-object in V.

$$c \in \mathcal{C} \Rightarrow \mathcal{C}[c] = \mathcal{C}(\cdot\,, c) \in V^{\mathcal{C}^{\mathrm{op}}}$$

Yoneda Lemma: $V = \mathrm{Set}$, $c \in \mathcal{C}$, $F \in \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \Rightarrow F(c) \cong \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(\subseteq, F)$. Also works in the enriched setting, at least for concrete categories, such as Vect.

Definition I.1. Let Γ be an abelian group, viewed as a discrete category enriched in Vect. A Γ-graded vector space is defined to be a functor $\Gamma \to \mathrm{Vect}$. Then Vect^{Γ} denotes the category of Γ-graded vector spaces. ■

- Each $V \in \mathrm{Vect}^{\Gamma}$ is given by a collection $V = (V_\gamma)_{\gamma\in\Gamma}$ and $\mathrm{Vect}^{\Gamma}(V, W) = \prod_{\gamma\in\Gamma} \mathrm{Vect}(V_\gamma, W_\gamma)$.
- For $\gamma \in \Gamma$ we thus have $\underline{\gamma} = \Gamma[\gamma] = \mathrm{Vect}^{\Gamma}(\cdot\,, \gamma) = \Bbbk$ concentrated in degree γ.
- By Yoneda, $V_\gamma = \mathrm{Vect}^{\Gamma}(\underline{\gamma}, V)$. Representable functors are otherwise known as *lines*.
- (note: $\Gamma(\gamma, \gamma') = \Bbbk$ iff $\gamma = \gamma'$ and 0 otherwise)

Tensor product:

$$(V \otimes W)_\gamma = \oplus_{\gamma'+\gamma''=\gamma} V_{\gamma'} \otimes W_{\gamma''}$$

Braiding

$$\epsilon : \Gamma \to \mathbb{Z}_2 \text{ pairity map },\quad \tau_{V,W} : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto (-1)^{\epsilon(\deg(v))\epsilon(\deg(w))} w \otimes v$$

If $\phi : \Gamma' \to \Gamma'$ is a group homomorphism, then we obtain functors

$$\mathrm{Vect}^{\Gamma} \underset{\varphi^*}{\overset{\varphi_!}{\rightleftarrows}} \mathrm{Vect}^{\Gamma}$$

$(\varphi_! V)_{\gamma'} = \oplus_{\gamma \in \varphi^{-1}(\gamma')} V_\gamma$. In particular, we obtain

$$\mathrm{Vect}^\Gamma \xrightarrow{\epsilon_!} \mathrm{Vect}^{\mathbb{Z}_2}$$

- $\mathrm{Vect}^\Gamma(V \otimes W, U) \cong \mathrm{Vect}^\Gamma(V, \underline{Vect}^\Gamma(W,U))$ with $\underline{\mathrm{Vect}}^\Gamma(W,U)_\gamma \cong \prod_{g'} \mathrm{Vect}(\mathrm{W}_{\gamma',\mathrm{U}_{\gamma'+\gamma}})$
- dual: $V^* = \underline{\mathrm{Vect}}^\Gamma(V, \underline{0})$
- shift: $V[\gamma] = \underline{\mathrm{Vect}}^\Gamma(\underline{\gamma}, V)$

Definition I.2. V is *finite-dimensional* ifV^{tot} is so. $\dim(V) = \sum_\gamma (\dim(V_\gamma))e_\gamma \in \mathbb{N}[\Gamma]$, and $\mathrm{sdim}(V) = \sum_\gamma (-1)^\epsilon(\gamma) \dim V_\gamma \in \mathbb{Z}$ ■

- typically, $\Gamma = \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z} \times \mathbb{Z}_2$
- a complex will always be a cochain complex, the differential always increases degrees by 1 (and is odd)
- $\mathbb{N}$-graded v.s. are considered as $\mathbb{Z}$-graded concentrated in deg ≥ 0. If V is $\mathbb{N}$-graded, then V^* is $-\mathbb{N}$-graded
- $\mathbb{N}$-grading $\Leftrightarrow$ action of a multiplicative monoid $\mathbb{M}(\Bbbk)$
- $\mathbb{Z}$-grading $\Leftrightarrow$ action of a multiplicative group $\mathbb{G}(\Bbbk)$ if $\Bbbk$ is algebraically closed of characteristic 0.
- $\mathbb{Z}_2$-grading $\Leftrightarrow$ action of $\mu_2 = \{\pm 1\}$.

I.2 Superdomains

Definition I.3. For $U \subseteq \mathbb{R}^n$ open, $U^{n|m} = (U, \mathcal{O}_{U^{n|m}})$, where $\mathcal{O}_{U^{n|m}}(U') = C^\infty(U') \otimes \Lambda[\xi^1, ..., \xi^m]$ as a $\mathbb{Z}_2$-graded commutative algebra. A morphism

$$U^{n|m} \to V^{n|m}$$

is just a map pf locally ringed spaces (parity-preserving), i.e. $f_0 : U \to V$ is continuous and $f^* : \mathcal{O}_V \to (f_0)_* \mathcal{O}_U$ ■

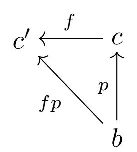
It can be shown that

- f_0 is C^∞
- f is determined by its values on coordinates

$$U^{n|m} \to V^{p|q}...$$

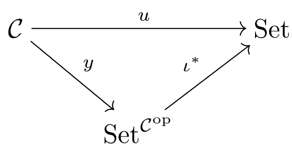
Definition I.4. A supermanifold is a locally ringed space $M = (M_0, \mathcal{O}_M)$, locally isomorphic to a superdomain. Denote the corresponding category by SM ■

Points: For $\mathcal{C}$ any category, denote by $Y : \mathcal{C} \to \mathrm{Set}$, $c \mapsto \mathcal{C}[c] = \mathcal{C}(\cdot, c) = \underline{c}$. $\mathcal{C}(a,b) \cong \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(\underline{a}, \underline{b})$ $\rightsquigarrow \mathcal{C}[c]$ is the functor of points on c. $b \xrightarrow{p} c$ is a "b-point" in $\mathcal{C}$



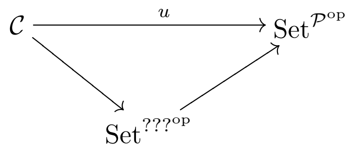
("fp is the value on f at the point p")

- This language of points can be extended to not necessarily representable presheaves
- if $* \in \mathcal{C}$ is terminal, then $\mathcal{C}$ is *concrete* if u is faithful



  for $\iota : \{*\} \hookrightarrow \mathcal{C}$

- SM is *not* concrete, but if $\mathcal{P} = \mathrm{Odd}_0 = \{(*, \Lambda[\xi^1, ..., \xi^m]\} \hookrightarrow \mathrm{SM}$, then



  is faithful

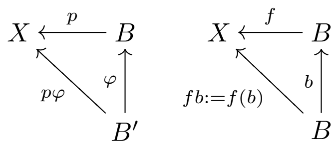
Uses of the functor of points:

- $\mathrm{SM}^{\mathrm{op}} \to \mathrm{Vect}$, $M \mapsto C^\infty(M)_0$ (resp. $M \mapsto C^\infty(M)_1$) are represented by $\mathbb{R}$ (resp. $\mathbb{R}^{0|1}$)

Corollary I.5. $\mathrm{Vect}_f^{\mathbb{Z}_2} \cong \mathrm{Vect}(\mathrm{SM})$ *(vector space objects in* SM*)*

II Courant algebroids and Courant-Dorfman algebras

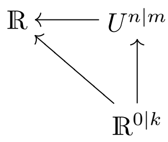
$\mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ $X \mapsto C[X] = \mathcal{C}(\cdot, X)$ (Joneda embedding)



(this always holds). At the other extreme, we have
For $* \in \mathcal{C}$, consider

$u(x) = \mathcal{C}(*, x)$. Then $\mathcal{C}$ is conctrete $\Leftrightarrow$ u is faithful
The intermediate step is: $* \subseteq \mathcal{P} \hookrightarrow \mathcal{C}$

E.g.: $\mathcal{C} = \mathrm{SM}$, $\mathcal{P} = \mathrm{Odd}_0 = \mathrm{Grass}_f^{\mathrm{op}}$ (connected odd super manifolds). For $f : U^{P|q} \to \mathbb{R}$, consider $y = f_0(x) + \frac{1}{2}\xi\xi f_2(x)$,



detects everything.

II.1 Even and odd tangent bundles

Let $M \in \mathrm{SM}$, $M \overset{p}{\leftarrow} B$ a point.

Definition II.1. An even (odd) derivation at p is a linear even (odd) map $D : C^\infty(M) \to C^\infty(B)$ satisfying

$$D(fg) = (Df)g(p) + (-1)^{\deg(D)\deg(f)} f(p)Dg.$$

With this we set $\mathrm{Der}_{0(1)}(M,p) = \{\text{all even (odd) derivations at } p\}$ and

$$\mathrm{Der}_{0(1)}(M)(B) = \coprod_{b:B\to M} \mathrm{Der}_{0(1)}(M,b).$$

This defines a presheaf $\mathrm{Der}_{0(1)}(M) : \mathrm{SM}^{\mathrm{op}} \to \mathrm{Set}$. ■

Definition II.2. An even tangent vector at p is a map $B \times \mathbb{R} \xrightarrow{\gamma} M/\sim$

- $\gamma|_{B\times\{0\}} = p$
- $\gamma \sim \gamma'$ first order equivalence

$\Rightarrow \mathrm{Tan}_0(M) : \mathrm{SM}^{\mathrm{op}} \to \mathrm{Set}$. An odd tangent vector at p is a map $B\times\mathbb{R}^{0|1} \xrightarrow{\gamma} M$ s.th. $\gamma|_{B\times\{0\}} = 0$ (this is just a family of odd curves parametrized by B). $\Rightarrow \mathrm{Tan}_1 : \mathrm{SM}^{\mathrm{op}} \to \mathrm{Set}$ ■

Proposition II.3. $\mathrm{Der}_{0(1)}(M)$, $\mathrm{Tan}_{0(1)}(M)$ *are isomorphic and representable by supermanifolds* TM *(resp.* ΠTM*). Moreover, these have a natural vector bundle structure s.th. the space of sections is the same as the space of even (resp. odd) vector fields on* M*.*

Corollary II.4. *The presheaf* $\underline{\mathrm{SM}}(\mathbb{R}^{0|1}, M)$ $B \mapsto \mathrm{SM}(B \times \mathbb{R}, M)$ *is representable by* $\Pi TM \cong M^{\mathbb{R}^{0|1}}$

Proposition II.5. $\underline{\mathrm{SM}}(X,Y)$ *is representable for all* $Y \Leftrightarrow X \in \mathrm{Odd}_c$ *(where the subscript* c *denotes finite unions of superpoints).*

II.2 Composition of internal homs

Want to define $\underline{C}(X,Z) \times \underline{C(Z,Y)} \to \underline{C}(X,Z)$. For $b \in \mathcal{C}$, define

$$C(Y \times b, Z) \times C(X \times b, Y) \to C(X \times b, Z), \quad (f,g) \mapsto f(g \times 1)(1 \times \Delta)$$

In particular, $\underline{C}(Y,Y)$ is a presheaf of monoids acting for all $X \in \mathcal{C}$ from the left on $\underline{C}(X,Y)$ and for all $Z \in \mathcal{C}$ from the right on $\underline{C}(Y,Z)$.

Corollary II.6. $\Pi TM \cong M^{\mathbb{R}^{0|1}}$ *has a natural smooth action of the monoid objects* ${\mathbb{R}^{0|1}}^{\mathbb{R}^{0|1}} \cong \mathbb{M}_m(\mathbb{R}) \rtimes \mathbb{R}^{0|1}$

Note that a right action of $\mathbb{M}_m(\mathbb{R}) \rtimes \mathbb{R}^{0|1}$ is the same as an $\mathbb{N}$-grading and an odd differential of degree $+1$.

$\Omega(M) \cong C^\infty(\Pi TM)$. Then the action is just the usual grading of differential forms, together with the de Rham differential.

Definition II.7. A differential $\mathbb{N}$-graded supermanifold is an $M \in \mathrm{SM}$ with a right action of ${\mathbb{R}^{0|1}}^{\mathbb{R}^{0|1}}$. A dg-manifold is the same such that (-1) acts as the parity involution. ■

Fact: a dg-manifold is the same thing as a locally ringed space $(M, \mathcal{O}_M)$ (in $\mathbb{N}$-graded commutative algebras over $\mathbb{R}$), which is locally isomorphic to $(U, \mathcal{O}_U)$ (+ a derivation of degree $+1$ and square 0), where

$$\mathcal{O}_U(U') = C^\infty(U') \otimes S(V^*)$$

where $U \subseteq \mathbb{R}^n$ is open and V is an $(-\mathbb{N})$-graded vector space.

III Presheaves etc.

Recall: A dg-manifold $(M_0, \mathcal{O}, d)$ is locally isomorphic to $(U \subseteq \mathbb{R}^n, \mathcal{O}_U = C_U^\infty \otimes S^\bullet V^*)$. This is equivalently given by a supermanifold $(M, \mathcal{O}_M)$, together with a right action of $\mathbb{R}^{0|1^{0|1}} = \mathbb{M}_m(\mathbb{R}) \rtimes \mathbb{R}^{0|1}$ (such that -1 acts as parity involution)

III.1 Some adjunctions:

- dgMan $\overset{u_{0!}}{\longleftarrow}$ Man, $u_{0!}(N) = N$ with zero grading and d
- dgMan $\overset{u_0}{\longrightarrow}$ Man, $u_0(N) = \Pi TN$ with the canonical grading and d the de Rham differential.
- dgMan $\overset{u_{0*}}{\longleftarrow}$ Man

The unit $M \mapsto u_{0*}u_0M = \Pi TM_0$ is called the *anchor map.*

- dgMan $\overset{u_!}{\longleftarrow}$ GrMan, $u_!(N) = N$ with zero d
- dgMan $\overset{u}{\longrightarrow}$ GrMan
- dgMan $\overset{u_*}{\longleftarrow}$ GrMan, $u_*(N) = T[1]N$ $(\deg(dx) = \deg(x) + 1)$

Using u_{0*}, one can define maps from a manifold N to a dg-manifold M to be a dg-map from $T[1]N$ to M. One can also define homotopies of such maps to be a dg-map $T[1](N \times I) \to M$ with specified boundary conditions. In fact, the category dgMan is naturally enriched in $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ w.r.t.

$$\mathrm{Map}(M, M')_n := \mathrm{dgMan}(M \times T[1]\Delta^n, M').$$

III.2 The tangent complex

$M = (M_0, \mathcal{O}_M, d)$, a linearized action of $\mathbb{R}^{0|1^{\mathbb{R}^{0|1}}}$ on the restricted bundle $T|_{M_0} M$ gives rise to a cochain complex

$$\mathbb{T}M = (T_{-n} \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_3} T_{-2} \xrightarrow{\delta_2} T_{-1} \xrightarrow{\delta_1 := p} T_0 = TM_0)$$

the integer n is called the *degree* of M (the highest degree of a local coordinate). Here δ in the dual on the linear part of d.
$\dim H^i(\mathbb{T}_x M, \delta)$ can vary with x, but $\mathrm{sd}(M) = \sum(-1)^{i \dim H^i} = \sum(-1)^i \dim T_i \in \mathbb{Z}$ does not.

Example III.1.

- H: a vector space $n > 0$, $H[n] = (\mathrm{pt}, S(H[n])^*) = (\mathrm{pt}, \rho^* h^*[-n])$. These are important for they play the role of Eilenberg–Mac Lane spaces for dg-manifolds, i.e.

$$\mathrm{dgMan}(T[1]N, H[n]) \cong \Omega^{n,\mathrm{cl}}(N, h)$$
$$\mathrm{dgMan}(T[1]N, H[n])/\mathrm{homotopy} \cong H^n_{\mathrm{dR}}(N, h)$$

- $\deg(M) = 1$,

$$M = A[1] = (M_0, S^\bullet(A[1])^*) = (M_0, S(A^*[-1])) = (M_0, \Lambda A^*)$$

  for $A \to M$ a vector bundle. A differential is the same thing as a Lie algebroid structure on A. ■

$\mathrm{Vect}(M)$ (differential operators on smooth functions) is a differential graded Lie algebra concentrated in degrees $[-n, \infty)$, $D = \{\cdot\,, d\}$

($n=1$): $\mathrm{Vect}^{-1}(A[1]) \cong \Gamma(A) \ni v \mapsto \iota_v$, $\{\iota_v, \iota_v\} = 0$. Then the derived bracket $\{\iota_v\{\iota_w, d\}\} = \iota_{[v,w]}$ (Cartan) defines a bracket on $\Gamma(A)$. $L^0 = Z^0(\mathrm{Vect}(M))$ infinitesimal automorphisms of (M,d), $L^{-i} = \mathrm{Vect}^{-i}(M)$ for $i > 0$, $L^{-n} \to \cdots \to L^{-1} \to L^0$ is also dgla

($n=1$): $L^{-1} \xrightarrow{D} L^0$ is a Lie algebra crossed module.

($n>1$): $\{\cdot\,,\cdot\}$ on $L^{-n} \to \cdots \to L^{-1} \to L^0$ does not vanish, $[\cdot\,,\cdot] = \{\cdot\,, d, \cdot\}$ is not skew-symmetric any more, so be obtain only a Leibnitz algebra. **Higher derived brackets:** $[v,w,u] := \pm\{v,\{w,\{u,d\}\}\}$ (and so on) (**Ex.:** A principal $\mathbb{R}[n]$-bundle over $T[1]M_0$ (infinitesimal $(n-1)$-gerbe), classified by $H^{n+1}(M)$).

($n=2$): $\mathrm{Vect}^{<0}(P) : \mathrm{Vect}^{-2}(P) \cong C^\infty(M_0)$ ($f\frac{\partial}{\partial t}$ for t a coordinate on $\mathbb{R}[2]$)

$$0 \to \Omega^1(M) \to \mathrm{Vect}^{-1}(P) := E \to \mathrm{Vect}(M) \to 0,$$

$\{\cdot\,,\cdot\} : E \otimes E \to C^\infty(M)$ and $[\cdot\,,\cdot] : E \otimes_{\mathbb{R}} E \to E$ a Lebnitz algebra + compatibility conditions. This defines a Courand algebroid and gives rise to "generalised differential geometry" in the sense of Hitchin and Gualtieri.