Diffeomorphism supergroups

Seminar Sophus Lie 07/2009

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Outline

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A *super vector space* V over a field $\mathbb{K}$ is simply a $\mathbb{Z}/2$-graded vector space:

$$V = V_{\bar{0}} \oplus V_{\bar{1}}$$

Morphisms of super vector spaces are linear maps preserving the grading.

Direct sums and tensor products of super vector spaces are simply those of $\mathbb{Z}/2$-graded spaces:

$$(V \otimes W)_{\bar{i}} = \bigoplus_{\bar{i}=\bar{j}+\bar{k}} V_{\bar{j}} \otimes W_{\bar{k}}$$

A $\mathbb{K}$-superalgebra A is a super vector space with a bilinear map

$$\mu : A \otimes_{\mathbb{K}} A \to A.$$

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Up to here, we have just rephrased $\mathbb{Z}/2$-graded algebra over a field. The difference is

Definition

The category $\mathrm{SVec}_{\mathbb{K}}$ of super vector spaces is the monoidal category of $\mathbb{Z}/2$-graded vector spaces endowed with the "super" braiding

$$\begin{aligned} c_{V,W} : V \otimes W &\xrightarrow{\sim} W \otimes V, \\ v \otimes w &\mapsto (-1)^{p(v)p(w)} w \otimes v, \end{aligned}$$

where $p(\cdot)$ denotes parity.

This redefines the notion of commutativity: whenever in a multiplicative expression we permute an odd element past another odd element, a factor (-1) is introduced.

More precisely, a map

$$\phi : V \otimes V \to W$$

of super vector spaces is *supersymmetric*, if $\phi = \phi \circ c_{V,V}$.

As an example, a superalgebra A is *supercommutative*, if

$$a \cdot b = (-1)^{p(a)p(b)} b \cdot a$$

holds for all homogeneous $a, b \in A$.

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Definition

A *Lie superalgebra L* is a superalgebra whose multiplication $[,]$ satisfies

- *anti-supersymmetry*: $[x, y] + (-1)^{p(x)p(y)}[y, x] = 0$
- *super Jacobi*: $[x, [y, z]] + (-1)^{p(x)(p(y)+p(z))}[y, [z, x]] + (-1)^{p(z)(p(x+p(y))}[z, [x, y]] = 0$

Representations of superalgebras are defined in the obvious way.
All of the above can be generalized from $\mathbb{Z}/2$-graded vector spaces to $\mathbb{Z}/2$-graded modules over $\mathbb{Z}/2$-graded rings.

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Examples

1. Every ordinary vector space V is a super vector space with $V_{\bar{1}} = 0$.
2. One has the standard super vector spaces $\mathbb{K}^{m|n}$ with m even and n odd dimensions.
3. The Grassmann algebra Λ_n is the free supercommutative algebra on n odd generators $\theta_1, \ldots, \theta_n$:

$$\theta_i \theta_j = -\theta_j \theta_i \quad \Rightarrow \quad \theta_i^2 = 0,$$

$\Rightarrow$ the monomials

$$\theta_{i_1} \cdots \theta_{i_j}, \quad i_1 < i_2 < \ldots < i_j$$

form a basis for Λ_n.

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form a basis for Λ_n.

We can then also write down polynomials in m even and n odd variables. Such a polynomial

$$f \in \mathbb{K}[x_1, \ldots, x_m, \theta_1, \ldots, \theta_n]$$

has the form

$$f = f_0(x) + \sum_i f_i(x)\theta_i + \sum_{i<j} f_{ij}(x)\theta_i\theta_j + \ldots$$

Thus,

$$\begin{aligned} \mathbb{K}[x_1, \ldots, x_m, \theta_1, \ldots, \theta_n] &\cong \mathbb{K}[x_1, \ldots, x_m] \otimes \Lambda_n \\ &\cong \mathbb{K}[x_1, \ldots, x_m][\theta_1, \ldots, \theta_n]. \end{aligned}$$

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A smooth superdomain $\mathcal{U}$ of dimension $m|n$ is given by

- an open domain $U \subset \mathbb{R}^m$
- the algebra of *superfunctions* $C^\infty(\mathcal{U}) = C^\infty(U)[\theta_1, \ldots, \theta_n]$, where the θ_i are odd generators

$U \subset \mathbb{R}^m$ with its algebra of smooth functions $C^\infty(U)$ is called the underlying domain of $\mathcal{U}$.

If $x_1 \ldots, x_m$ are coordinates on U, then every superfunction on $\mathcal{U}$ can again be written as

$$f = f_0(x) + \sum_i f_i(x)\theta_i + \sum_{i<j} f_{ij}(x)\theta_i\theta_j + \ldots$$

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A morphism of superdomains is a morphism of its superalgebras of functions. If $\mathcal{U}$ is a superdomain with coordinates $x_1, \ldots, \theta_n$ and $\mathcal{V}$ is a superdomain with coordinates $y_1, \ldots, y_k, \eta_1, \ldots, \eta_l$ then a morphism $\phi : \mathcal{U} \to \mathcal{V}$ is given by a homomorphism

$$\phi^* : C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U}).$$

One can show that such a morphism is completely determined by its action on the coordinates, i.e., it can be written as

$$\begin{aligned} y_i &\mapsto g_i^0(x) + g_i^{\alpha\beta}(x)\theta_\alpha\theta_\beta + \ldots \\ \eta_j &\mapsto h_j^\alpha(x)\theta_\alpha + h_j^{\alpha\beta\gamma}(x)\theta_\alpha\theta_\beta\theta_\gamma + \ldots \end{aligned}$$

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Definition

A smooth supermanifold $\mathcal{M} = (M, C^\infty_\mathcal{M})$ of dimension $m|n$ consists of a topological space M together with a sheaf $C^\infty_\mathcal{M}$ of supercommutative $\mathbb{R}$-algebras which is locally isomorphic to smooth superdomains of dimension $m|n$.

Morphisms of supermanifolds are morphisms of ringed spaces. In fact one can show that a morphism $\phi : \mathcal{M} \to \mathcal{N}$ is completely determined by its action on global sections of the structure sheaf. More precisely, one has

$$\mathrm{Hom}_{\mathsf{SMan}}(\mathcal{M}, \mathcal{N}) \cong \mathrm{Hom}_{\mathsf{SAlg}}(C^\infty_\mathcal{N}(\mathcal{N}), C^\infty_\mathcal{M}(\mathcal{M})).$$

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Examples

- Every super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ defines a linear supermanifold $\mathcal{V} = (V_{\bar{0}}, C^\infty_{V_{\bar{0}}} \otimes \wedge^\bullet V^*_{\bar{1}})$.
- In this way every purely odd space $\mathbb{R}^{0|n}$ defines a "superpoint"

$$\mathcal{P}_n = (\{*\}, \Lambda_n)$$

- More generally, given a smooth manifold M and a vector bundle $E \to M$, one defines a supermanifold

$$\Pi E = (M, \wedge^\bullet E^*),$$

where E^* is the dual bundle and $\wedge^\bullet E^*$ is the sheaf of sections of the exterior algebra over E^*. That is, we have declared E to consist of "odd dimensions".

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The naive concept of points is insufficient for supermanifolds, just as for any nonreduced variety.

For an ordinary manifold M, a point can be viewed as a map of the one-point manifold $(\{*\}, \mathbb{R})$ into M.

But for a supermanifold $\mathcal{M}$, a morphism $\phi : (\{*\}, \mathbb{R}) \to \mathcal{M}$ amounts to an evaluation map

$$\mathrm{ev}_p : C^\infty_{\mathcal{M},p} \to \mathbb{R}$$

that associates with the germs of superfunctions at $p \in M$ a number. Since $\mathbb{R}$ has no nilpotents, that quotients out the nilpotent ideal $N_\mathcal{M} \subset C^\infty_\mathcal{M}$.

$\Rightarrow$ the ordinary "points" only reproduce the underlying manifold of $\mathcal{M}$.

Analogously one concludes that superfunctions can not be described as mere maps $\mathcal{M} \to \mathbb{R}$.

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This is a well-known problem in algebraic geometry.
Solution: use the *functor of points* furnished by the Yoneda lemma. That is, instead of working with a supermanifold $\mathcal{M}$, we may work with the functor $\mathrm{Hom}_{\mathsf{SMan}}(-, \mathcal{M}) : \mathsf{SMan}^{op} \to \mathsf{Sets}$.
The assignment $\mathcal{M} \mapsto \mathrm{Hom}(-, \mathcal{M})$ is a fully faithful embedding.
Given another supermanifold B, the set $\mathcal{M}(B) := \mathrm{Hom}(B, \mathcal{M})$ is called the B-points of $\mathcal{M}$. One may view them as the sections of the trivial bundle $B \times \mathcal{M} \to B$.
A morphism $\phi : \mathcal{M} \to \mathcal{N}$ gets replaced by a functorial set of maps $\phi_B : \mathcal{M}(B) \to \mathcal{N}(B)$.
It turns out that it is sufficient to evaluate the functors of points on the superpoints $\mathcal{P}_n = (\{*\}, \Lambda_n)$, $n = 0, 1, 2, \ldots$.

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Definition

A (Lie) supergroup $\mathcal{G}$ is a group object in the category of supermanifolds.

There are two equivalent ways to say what that means:

1. Each set of points $\mathcal{G}(B)$ is a group and each morphism $\psi : B \to B'$ induces a group homomorphism $\mathcal{G}(\psi) : \mathcal{G}(B') \to \mathcal{G}(B)$.
2. There are morphisms $m : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $i : \mathcal{G} \to \mathcal{G}$ and $e : \mathcal{P}_0 \to \mathcal{G}$ which satisfy certain diagrams encoding the axioms of a group.

So, a Lie supergroup is not a group in the ordinary sense, it is not even possible to describe it as a single set with some structure.
However, the underlying manifold $G = \mathcal{G}(\mathcal{P}_0)$ of a supergroup $\mathcal{G}$ is an ordinary Lie group in a natural way.

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Diffeomorphism supergroups

Given a supermanifold $\mathcal{M}$, its automorphisms only form an ordinary group $\mathrm{Aut}(\mathcal{M})$.

$\Rightarrow$ we really want a diffeomorphism *supergroup* $\mathcal{SD}iff(\mathcal{M})$ of which $\mathrm{Aut}(\mathcal{M})$ would only form the underlying group.

Such a supergroup would have to be a subobject of an *inner Hom object*, i.e., of a supermanifold of maps $\mathcal{M} \to \mathcal{M}$.

Such an inner Hom object is defined by the adjunction formula

$$\mathrm{Hom}(\mathcal{B}, \underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{M})) \cong \mathrm{Hom}(\mathcal{B} \times \mathcal{M}, \mathcal{M}),$$

i.e., it is an object representing a certain functor $\mathrm{SMan}^{op} \to \mathrm{Sets}$.

The diffeomorphism supergroup would consist of the invertible maps in each of the sets of $\mathcal{B}$-points $\mathrm{Hom}(\mathcal{B} \times \mathcal{M}, \mathcal{M})$.

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As mentioned above, it is enough to study the cases where $\mathcal{B} = \mathcal{P}_0, \mathcal{P}_1, \ldots$. The $\mathcal{P}_n$-points $\mathrm{Hom}(\mathcal{P}_n \times \mathcal{M}, \mathcal{M})$ of the inner Hom object are nothing but invertible morphisms of families

$$\begin{array}{ccc} \mathcal{P}_n \times \mathcal{M} & \longrightarrow & \mathcal{P}_n \times \mathcal{M} \\ & {\scriptstyle p_1} \searrow \quad \swarrow {\scriptstyle p_1} & \\ & \mathcal{P}_n & \end{array}$$

$\mathrm{Aut}(\mathcal{M})$ corresponds to morphisms over the ordinary point $\mathcal{P}_0 = (\{*\}, \mathbb{R})$.
In fact,

$$C^\infty(\mathcal{P}_n \times \mathcal{M}) \cong C^\infty(\mathcal{M}) \otimes_{\mathbb{R}} \Lambda_n.$$

This means the “higher” points of $\mathcal{SDiff}(\mathcal{M})$ are morphisms $\mathcal{M} \to \mathcal{M}$ involving additional odd parameters which are furnished by the odd coordinates of $\mathcal{P}_n$.

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Let us investigate the case of the $\mathcal{P}_1$ points to get a feeling for those higher points.
A $\mathcal{P}_1$-point of $\mathcal{SD}iff(\mathcal{M})$ is a morphism $\phi : \mathcal{P}_1 \times \mathcal{M} \to \mathcal{M}$, which in turn is given by a morphism of superalgebras

$$\phi^* : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}) \otimes \Lambda_1.$$

Writing $\Lambda_1 = \mathbb{R}[\theta]$ we see that for a superfunction $f \in C^\infty(\mathcal{M})$ we have

$$\phi^*(f) = \alpha_0(f) + \alpha_1(f)\theta,$$

where α_0, α_1 are linear maps of which α_0 preserves parity, while α_1 reverses it (since ϕ has to preserve parity).
Now we need to enforce

$$\phi^*(fg) = \phi^*(f)\phi^*(g) = [\alpha_0(f) + \alpha_1(f)\theta] \cdot [\alpha_0(g) + \alpha_1(g)\theta]$$

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This, in turn, is equivalent to

$$\begin{aligned} \alpha_0(fg) &= \alpha_0(f)\alpha_0(g) \\ \alpha_1(fg) &= (-1)^{p(g)}\alpha_1(f)\alpha_0(g) + \alpha_0(f)\alpha_1(g) \end{aligned}$$

So α_0 is just an endomorphism of $C^\infty(\mathcal{M})$ while α_1 is a so-called derivation over α_0, i.e., it can be written as

$$\alpha_1(f) = X \circ \alpha_0(f)$$

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One can show that the requirement that ϕ^* be invertible is equivalent to α_0 being invertible.
$\Rightarrow$ α_0 determines an ordinary automorphism $\mathcal{M} \to \mathcal{M}$
$\Rightarrow$ as a set, $\mathcal{SD}iff(\mathcal{M})(\mathcal{P}_1) \cong \mathrm{Aut}(\mathcal{M}) \times \mathcal{X}(\mathcal{M})_{\bar{1}}$, where $\mathcal{X}(\mathcal{M})_{\bar{1}}$ is the vector space of odd vector fields on $\mathcal{M}$.
The family viewpoint makes this result plausible: the existence of the odd parameter θ makes it possible to generate diffeomorphisms from odd vector fields, but these diffeomorphisms are "infinitesimal" since θ squares to zero.
This picture is consistent for all $\mathcal{P}_n$-points of $\mathcal{SD}iff(\mathcal{M})$: each of those is determined by an automorphism $\phi_0 : \mathcal{M} \to \mathcal{M}$ and 2^{n-1} odd and $2^{n-1} - 1$ even vector fields on $\mathcal{M}$.
How to turn these sets of points into manifolds and to exhibit these manifolds as the points of an (infinte-dimensional) supermanifold $\mathcal{SD}iff(\mathcal{M})$ is another story...

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