Turaev-Viro Theory and Extended TQFTs

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Outline

Introduction

- Turaev-Viro invariants were defined in 1993 by Turaev and Viro using quantum 6j-symbols and generalized by Barrett-Westbury to an arbitrary spherical fusion category.
- If $\mathcal{C}$ is a modular category, it was shown by Turaev that $Z_{TV,\mathcal{C}}(\mathcal{M}) = Z_{RT,\mathcal{C}}(\mathcal{M})Z_{RT,\mathcal{C}}(\overline{\mathcal{M}})$

We want to establish:

Theorem

If $\mathcal{C}$ is a spherical fusion category, $Z_{TV,\mathcal{C}}(\mathcal{M}) = Z_{RT,Z(\mathcal{C})}(\mathcal{M})$

To do this, we will describe Turaev-Viro theory as an extended theory.

Introduction$_2$

Definition

A (3-dimensional) TQFT is a symmetric monoidal functor $Z : Cob_3 \to Vec$

Let Σ be a closed surface and M be a 3-manifold possibly with boundary. Then the above definition gives

- $Z(\Sigma) \in Vec$ is a vector space
- $Z(M) \in Z(\partial M)$ a vector.

In particular, if $\mathcal{M}$ is closed, $Z(M) \in \mathbb{C}$

TV_1

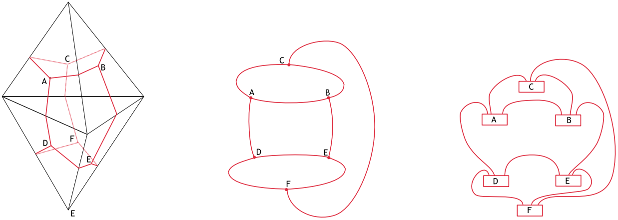
Throughout this section, $\mathcal{C}$ is a spherical fusion category. Let Σ be a triangulated closed surface, $\mathcal{M}$ a compact manifold, possibly with boundary. We construct $Z(\Sigma)$ as follows:

- Orient each edge of the triangulation
- Choose a labeling of edges $l : Edges(\mathcal{M}) \to Irr(\mathcal{C})$,
- For each 2-cell C define
  $H(\Sigma, l) = Hom(1, l(e_1) \otimes ... \otimes l(e_n))$
- Define $\mathcal{H}(\Sigma) = \bigoplus_{l} \bigotimes_{C} H(C, l)$, the "state-space".

  Note: The state-space clearly depends on the triangulation.

TV_2

For M a triangulated 3-manifold with boundary, $F \in M$ a 3-cell, we define $Z(F, l) \in \mathcal{H}(\partial F, l)$ as follows:



- Consider the dual graph to the triangulation
- Topologically, this defines a graph on $\partial F \simeq \partial D^3 = S^2$
- Label vertices with appropriate morphisms (equivalently, pick a vector $v \in \mathcal{H}(\partial F)^*$)
- By the results of Reshetikhin-Turaev, one can evaluate this graph to get a number.

TV_3

Finally, we define

$$Z_{TV}(\mathcal{M}) = \mathcal{D}^{-2v(\mathcal{M})} \sum_{l} (Z_{TV}(\mathcal{M}, l) \prod_{e} d_{l(e)}^{n_e})$$

This is not quite a TQFT as
$A = Z(\Sigma \times [0,1]) : \mathcal{H}(\Sigma) \to \mathcal{H}(\Sigma) \neq Id_{\mathcal{H}(\Sigma)}$.

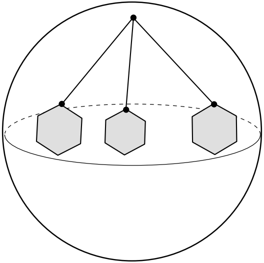
- $A^2 = A$, i.e. A is a projector
- Define $Z_{TV}(\Sigma) \equiv Im(A)$. Then $Z_{TV}(\Sigma)$ does not depend on triangulation of Σ
- Z_{TV} is functorial and thus defines a TQFT.

Polytope Decomposition

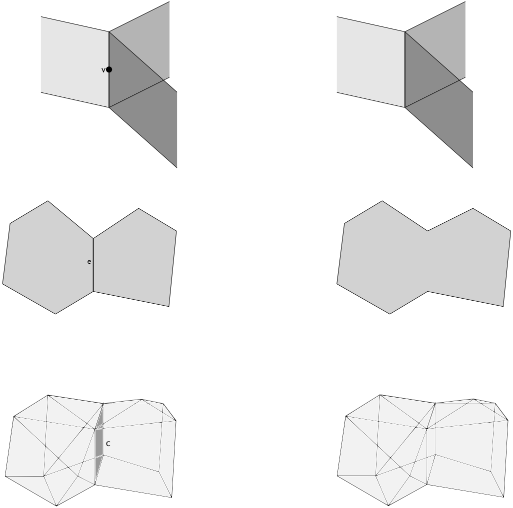
To extend TV to manifolds with corners, it is convenient to introduce the following modification.

Definition

A combinatorial 3-manifold is a 3-manifold with cell decomposition that may be obtained from a triangulation by a finite sequence of $\mathcal{M}_1 - \mathcal{M}_3$



Combinatorial Moves



TV for Polytope Decompositions

Theorem

The TV construction works for combinatorial manifolds.

Proof (sketch).

- The Pachner moves may be obtained by a sequence $\mathcal{M}_1 \to \mathcal{M}_3$
- The TV state sum is invariant under $\mathcal{M}_1 \to \mathcal{M}_3$

□

Preliminaries: Drinfeld Center

Let $\mathcal{C}$ be a spherical fusion category.

Definition

The Drinfeld Center $\mathcal{Z}(\mathcal{C})$ is the category with

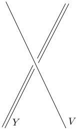
- Objects are pairs (Y, φ_Y) where $Y \in Obj(\mathcal{C})$ and $\varphi_Y : Y \otimes - \to - \otimes Y$ is a choice of natural isomorphism. This isomorphism must satisfy the (half)- braiding condition.
- Morphisms are morphisms in $\mathcal{C}$ that commute with the (half)-braiding condition.

Theorem (Mueger)

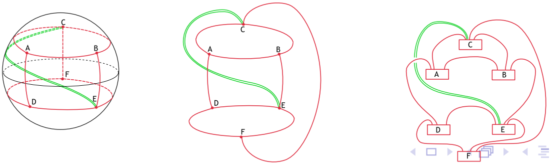
$Z(\mathcal{C})$ is a modular category. In particular, it is semisimple and braided with finitely many simple objects

Preliminaries: Drinfeld Center$_2$

We represent φ_Y graphically by a crossing. Note: strands labeled with objects from $Z(C)$ are doubled.



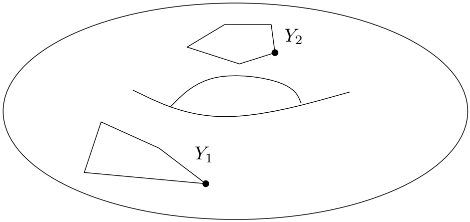
We can now consider graphs on S^2 and in the plane as before and add an edge, labeling it with an element of $Z(\mathcal{C})$ Again, using the theory of ribbon graphs, such a diagram will give us an element of **k**.



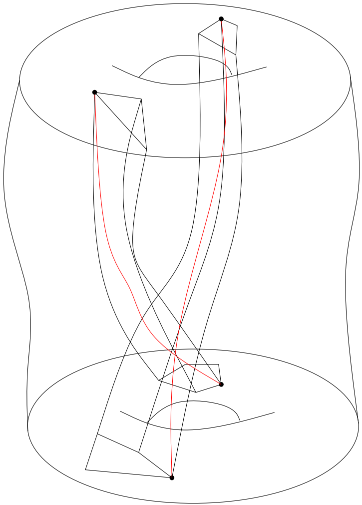
Preliminaries: Extended Surfaces

Definition

- An *extended surface* is a PL surface with a finite disjoint collection of embedded disks, each with a point fixed on the boundary
- A *coloring* of an extended surface Σ is a labeling of each embedded disk by an object of $Z(\mathcal{C})$



Preliminaries: Extended Manifolds



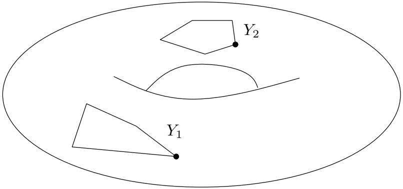
TV as an Extended Theory

We will now describe the Turaev-Viro construction as a (3-2-1) Extended TQFT. As before, we start with a spherical fusion category $\mathcal{C}$. By general theory (e.g. Lurie) we should assign

- To a closed 1-manifold a category. In our case, $Z_{TV}(S^1) = Z(\mathcal{C})$ and $Z(\emptyset) = Vec$
- To a surface Σ with boundary, a functor between categories. If Σ is closed, this will give us a functor $Vec \to Vec$, i.e. a vector space.

TV as an Extended Theory$_2$

It is convenient to view the surface as a cobordism from a union of circles to $\emptyset$.



The functor we get (for $\Sigma = S^2$) is simply $Hom(\mathbf{1}, - \otimes -)$. Note, If we color the extended surface with objects Y_1, Y_2, we get $Hom(\mathbf{1}, Y_1 \otimes Y_2)$, a vector space.

Extended Combinatorial 3-Manifolds

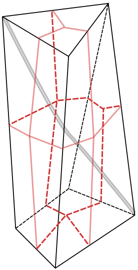
Definition

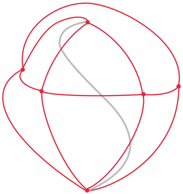
A *combinatorial extended 3-manifold* $\mathcal{M}$ is an extended PL manifold with a polytope decomposition such that

- The interior of an open tube is a single 3-cell of the decomposition. Additionally, each component of the tube's boundary is a single 2-cell in $\partial\mathcal{M}$ and the marked points on the boundaries of these 2-cells are vertices of the decomposition.
- To allow flexibility in gluing, we also allow *closed* tubes, i.e. embeddings of $S^1 \times D^2$ into $\mathcal{M}$. The requirements are similar to the open tube case

"Ribbon" Graphs from Tubes

Now Let $\mathcal{M}$ be a coloring of an extended combinatorial 3-manifold. Let T be a tube colored by $Y \in Z(\mathcal{C})$. Since $T \cong D^3$, $\partial T \cong S^2$. we once again get a graph on S^2 with an added strand crossing other edges.





As before, if we color all edges and vertices, we get a number.

3-2-1 State Sum

Finally, we define Z_{TV} for extended manifolds:

$$Z_{TV}(\mathcal{M}, \{Y_\alpha\}) = \mathcal{D}^{-2v(\mathcal{M})} \sum_{l} (Z_{TV}(\mathcal{M}, \{Y_\alpha\}, l) \prod_{e} d_{l(e)}^{n_e})$$

Theorem

Z_{TV} defines an extended TQFT. In particular,

- *Z_{TV} satisfies the gluing axiom both at the level of 3-manifolds and surfaces*
- *If $\mathcal{M}'$ and $\mathcal{M}''$ are two polytope decompositions of an extended PL 3-manifold $\mathcal{M}$ coinciding on the boundary, then $Z_{TV}(\mathcal{M}', \{Y_\alpha\}) = Z_{TV}(\mathcal{M}'', \{Y_\alpha\})$*
- *The cylinder is a projector, and as before defines for a surface Σ a vector space independent of decomposition.*

Proof of Theorem

We prove the statement

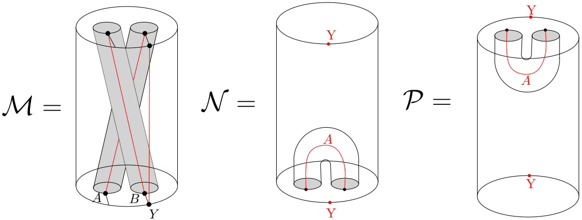
Theorem

As (3-2-1) TQFTs, $Z_{TV,\mathcal{C}} \cong Z_{RT,Z(\mathcal{C})}$

1. For a closed 3-manifold (and 3-manifold with boundary).
   - Do this for S^3 with an embedded link L inside.
   - Prove a *surgery* formula for TV theory.
2. For a closed surface (and surface with boundary), we need to show that there is a *natural* isomorphism between the corresponding vector spaces.

Together with the gluing axioms, this proves the theorem.

Computations



- $Z_{TV,\mathcal{C}}(\mathcal{M}) = Id_Y \otimes \sigma_{AB} \cong Z_{RT,Z(\mathcal{C})}(\mathcal{M})$
- $Z_{TV,\mathcal{C}}(\mathcal{N}) = Id_Y \otimes ev_A \cong Z_{RT,Z(\mathcal{C})}(\mathcal{N})$
- $Z_{TV,\mathcal{C}}(\mathcal{P}) = Id_Y \otimes coev_A \cong Z_{RT,Z(\mathcal{C})}(\mathcal{P})$

$\Rightarrow$ By gluing this is true for S^3 with a link inside, or more generally, $S^2 \times I$ with a tangle inside.

Surgery Formula

Let $\mathcal{M}$ and $\mathcal{N}$ be 3-manifolds with $\partial M \cong \Sigma \cong \partial N$.

Lemma (Gluing axiom)

Let $\varphi : \partial\mathcal{M} \to \partial\overline{\mathcal{N}}$ be a homeomorphism. By TQFT properties, φ induces an isomorphism $\varphi_* : Z(\partial\mathcal{M}) \to Z(\partial\overline{\mathcal{N}}) \cong Z(\partial\mathcal{N})^*$. Denote the result of gluing $\mathcal{M}$ and $\mathcal{N}$ via φ by $\mathcal{M} \sqcup_\varphi \mathcal{N}$. Then

$$Z_{TV,\mathcal{C}}(\mathcal{M} \sqcup_\varphi \mathcal{N}) = < \varphi_* Z(\partial\mathcal{M}), Z(\partial\mathcal{N}) >$$

Lemma (Surgery formula)

Let $\mathcal{M}_L = S^3$ with a framed link inside it. Let $M_{L'}$ be the result of performing surgery along a single component of L. Then

$$Z_{TV,\mathcal{C}}(M_{L'}) = \frac{1}{\mathcal{D}^2} Z_{TV,\mathcal{C}}(M_L)$$

Equivalence of (2,1) theories

We want to demonstrate a canonical isomorphism $Z_{TV,\mathcal{C}}(\Sigma) \cong Z_{RT,Z(\mathcal{C}}(\Sigma)$ for a compact surface Σ possibly with (colored) boundary.

- This issue arises when trying to show that the Reshetikhin-Turaev theory is well-defined. Related to choice made in pants decompositions of surfaces.
- Solved by Bakalov-Kirillov based on ideas of Moore-Seiberg. Uses so-called "surface parametrizations"
- There is a canonical *map* between this construction and the Turaev-Viro construction. Surface paramatrizations $\cong$ cell decompositions.

Fully Extended Theories

We want to describe TV theory as a fully extended TQFT. Such a theory should be completely described by a fully dualizable object X in some 3-category, which we call $\mathcal{M}$.

- Objects are monoidal categories
- 1-morphisms are bimodule categories
- 2-morphisms are bimodule functors
- 3-morphisms are functorial maps between functors

In $\mathcal{M}$, fully dualizable objects should correspond to spherical fusion categories.

In this setup,

- $Z(pt) = \mathcal{C}$, our original category.
- $Z(interval) = \mathcal{C}$ viewed as a $\mathcal{C} - \mathcal{C}$ bimodule category.
- $Z(S^1) = Z(\mathcal{C})$

And further, this reconstructs the (3,2,1) theory described in this talk (Work in progress).

Questions

- Can RT theory be described as a fully extended theory? If so, is there a nice relationship to TV theory?
- Can we get other flavors of 3D tqfts from state sums (Unoriented, Spin)?
- Does a good algebraic formalism exist to construct 4D state-sum theories and are there any useful examples?