Compactly Generated Stacks

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$\mathscr{CG}\mathfrak{TSt}$- Convenient 2-category of topological stacks:

- Cartesian Closed
- Complete

Remark. Some conventions:

- All spaces are assumed to be in the category of compactly generated Hausdorff spaces = $\mathbb{CGH}$.
- Will ignore all size issues.

I. Topological Stacks

A. Stacks

Let $\mathscr{C}$ be a small category. Let Gpd denote the 2-category of groupoids and let

$$Gpd^{\mathscr{C}^{op}} := \left\{ \begin{array}{l} \text{weak contravariant functors} \\ \mathscr{X} : \mathscr{C}^{op} \to Gpd \end{array} \right\}$$

denote the 2-category of weak presheaves in groupoids.

Remark. For any $\mathscr{X}$, $\mathscr{Y}$ in $Gpd^{\mathscr{C}^{op}}$, $\mathrm{Hom}\left(\mathscr{X}, \mathscr{Y}\right)$ is a groupoid.

Let J be a Grothendieck topology on $\mathscr{C}$. A **J-stack** is $\mathscr{X} \in Gpd^{\mathscr{C}^{op}}$ which satisfies descent for J. J-stacks form a sub-2-category of $Gpd^{\mathscr{C}^{op}}$ denoted by $\mathrm{St}_J\left(\mathscr{C}\right)$. The inclusion has a left 2-adjoint:

$$\mathrm{St}_J\left(\mathscr{C}\right) \underset{i}{\overset{St_J}{\rightleftarrows}} Gpd^{\mathscr{C}^{op}} ,$$

$\mathrm{St}_J \dashv i$, given by the "stackification" 2-functor.

B. Topological Stacks

Let $\mathscr{C} = \mathbb{CGH}$ and $J =$ open cover topology.
Let $TGpd =$ 2-category of topological groupoids.

There is an extended Yoneda 2-functor:

$$\begin{array}{rcl} \tilde{y} : TGpd & \rightarrow & Gpd^{\mathbb{CGH}^{op}} \\ \tilde{y}(\mathcal{G})(T) & := & \mathrm{Hom}_{TGpd}\left(T^{id}, \mathcal{G}\right), \end{array}$$

where T^{id} is the topological groupoid with objects T with only identity arrows.
Let $[\mathcal{G}] := \mathrm{St}(\tilde{y}(\mathcal{G})) \in \mathrm{St}(\mathbb{CGH})$. This is the topological stack associated to $\mathcal{G}$.

Definition 1. The 2-category $\mathfrak{TSt}$ of topological stacks is the essential image of $St \circ \tilde{y}$.

Intrinsic definition:

$\mathscr{X} \in \mathrm{St}(\mathbb{CGH})$ is a topological stack $\Leftrightarrow \exists$ a representable epimorphism, called an **atlas** $X \twoheadrightarrow \mathscr{X}$ from a space X. For $\mathscr{X} \cong [\mathcal{G}]$, $\mathcal{G}_0 \twoheadrightarrow [\mathcal{G}]$ is an atlas.

Explicitly: $[\mathcal{G}](T) \cong \mathrm{Bun}_{\mathcal{G}}(T) =$ groupoid of principal $\mathcal{G}$-bundles over T,

where a principal $\mathcal{G}$-bundle is a space $P \xrightarrow{\pi} T$ over T together with an action of $\mathcal{G}$ along the fibers of π

$$\begin{array}{ccc} \mathcal{G}_1 & \curvearrowright & P \\ \downarrow\downarrow & \overset{\mu}{\swarrow} & \downarrow \pi \\ \mathcal{G}_0 & & T \end{array}$$

such that

- π has local sections
- action is principal:

$$\begin{array}{rcl} \mathcal{G}_1 \times_{\mathcal{G}_0} P & \xrightarrow{\sim} & P \times_T P \\ (g,p) & \mapsto & (gp,p) \end{array}$$

In fact, $\mathfrak{TSt} \cong \mathrm{Bun}\, TGpd =$ bicategory of topological groupoids with left principal bibundles as morphisms.

II. Cartesian Closed Categories

Recall: A category $\mathscr{C}$ with binary products is called **Cartesian closed** if

- $\mathscr{C}$ has a terminal object
- For any two objects C and D, there exists $D^C \in \mathscr{C}_0$ such that for all objects E, there is a bijection
$$\mathrm{Hom}\left(E, D^C\right) \xrightarrow{\sim} \mathrm{Hom}(E \times C, D)$$
natural in E,D,C...

e.g.

- $\mathscr{C} = \mathrm{Set}$
- $\mathscr{C} = \mathrm{Set}^{\mathscr{D}^{op}}$, $G^F(C) := \mathrm{Hom}(C \times F, G)$
- $\mathscr{C} = \mathbb{CGH}$, with $X^Y = \mathbb{M}\mathrm{ap}(Y, X) =$ space of maps.

Want to generalize this last example to topological stacks.

III. Mapping Stacks

Would like for two topological stacks $\mathscr{X}$, $\mathscr{Y}$ to have a topological stack $\mathbb{M}\mathrm{ap}(\mathscr{Y}, \mathscr{X})$ such that for all topological stacks $\mathscr{Z}$, there is an equivalence of groupoids

$$\mathrm{Hom}(\mathscr{Z}, \mathbb{M}\mathrm{ap}(\mathscr{Y}, \mathscr{X})) \xrightarrow{\sim} \mathrm{Hom}(\mathscr{Z} \times \mathscr{Y}, \mathscr{X})$$

natural in $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{Z}$...
$\mathbb{M}\mathrm{ap}(\mathscr{Y}, \mathscr{X})$ is the "stack of maps from $\mathscr{Y}$ to $\mathscr{X}$".

Remark. If T is a space and G is a topological group (regarded as a topological groupoid with one object), by the 2-Yoneda lemma

$$\mathrm{Hom}(T, [G]) \xrightarrow{\sim} \mathrm{Bun}_G(T),$$

so $\mathbb{M}\mathrm{ap}(T, [G]) =$ moduli stack of principal G-bundles over T.

The larger 2-category $\mathrm{St}(\mathbb{CGH})$ of all stacks is Cartesian closed:

$$\mathbb{M}\mathrm{ap}(\mathscr{Y}, \mathscr{X})(T) := \mathrm{Hom}(T \times \mathscr{Y}, \mathscr{X}),$$

and it has all weak limits.

Problem: $\mathbb{M}\mathrm{ap}(\mathscr{Y}, \mathscr{X})$ may not be equivalent to $[\mathcal{G}]$ for any topological groupoid $\mathcal{G}$. Similarly for limits.
Partial Progress:

i) (B. Noohi '08) If $\mathscr{X}$ and $\mathscr{Y}$ are topological stacks and $\mathscr{Y} \cong [Y]$ with Y a topological groupoid with Y_0 and Y_1 compact, then $\mathbb{M}\mathrm{ap}(\mathscr{Y}, \mathscr{X})$ is a topological stack.

ii) $\mathbb{M}\mathrm{ap}(\mathscr{Y}, \mathscr{X})$ is *nearly* topological (paratopological) if:

   a) (B. Noohi '08) Y_0 and Y_1 are locally compact
   b) (D. C. '09) Y_0 is a CW-complex.

IV. Main Result

There is a 2-category $\mathscr{CG}\mathfrak{TSt}$, called **compactly generated stacks**, which:

1) Is Cartesian closed and complete
2) Is "essentially the same" as $\mathfrak{TSt}$:

   a) $\mathscr{CG}\mathfrak{TSt}$ contains $\mathbb{CGH}$ as a full subcategory
   b) $\mathscr{CG}\mathfrak{TSt}$ is a full sub-2-category of $\mathrm{St}(\mathbb{CGH})$.
   c) Given $\mathcal{G} \in TGpd \rightsquigarrow [\mathcal{G}]_{\mathscr{CG}} \in \mathscr{CG}\mathfrak{TSt}$ and all compactly generated stacks are of this form.

d) For every locally compact space X

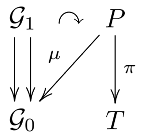
$$[\mathcal{G}]\,(X) \cong [\mathcal{G}]_{\mathscr{CG}}\,(X)$$

e) $[\mathcal{G}]$ and $[\mathcal{G}]_{\mathscr{CG}}$ are weakly homotopy equivalent.

F) $\mathscr{CG}\mathfrak{TSt}$ is (non-trivially) equivalent to the full sub-2-category of $\mathfrak{TSt}$ consisting of those topological stacks which admit a locally compact atlas.

Remark. F) is really strange! The only spaces in $\mathfrak{TSt}$ which admit a locally compact atlas are locally compact spaces. Since $\mathscr{CG}\mathfrak{TSt}$ contains all compactly generated spaces, under the equivalence of F), spaces in $\mathscr{CG}\mathfrak{TSt}$ are not sent to spaces!

Quick Definition:
Let $\mathcal{G} \in TGpd$ and T a space. A **compactly generated** principal $\mathcal{G}$ bundle over T is the same thing as an ordinary principal $\mathcal{G}$-bundle:



except π need only admit local sections over each compact subset of T (hence the terminology compactly generated).

$[\mathcal{G}]_{\mathscr{CG}}\,(T) \cong \mathrm{Bun}_{\mathcal{G}}^{\mathscr{CG}}\,(T)$ = groupoid of compactly generated principal $\mathcal{G}$-bundles over T.
In fact, $\mathscr{CG}\mathfrak{TSt} \cong$ a bicategory of topological groupoids with compactly generated bibundles as morphisms.
Compactly generated stacks have a more elegant definition, but first:

V. Why is $\mathbb{CGH}$ Cartesian closed?

1. In $\mathbb{TOP}$, if K is compact Hausdorff, then for any space X, $\mathbb{M}\mathrm{ap}\,(K, X)$- the space of maps endowed with the compact-open topology- serves as an exponential object.
2. A Hausdorff $Y \in \mathbb{TOP}$ is compactly generated if and only if $Y = \varinjlim_{K_\alpha \hookrightarrow Y} K_\alpha$ = colimit of all its compact subsets.

i) For all X, $\mathbb{M}\mathrm{ap}\,(K_\alpha, X)$ exists

ii) $\mathbb{CGH}$ has all limits

So by general properties of limits and colimits, the space

$$\mathbb{M}\mathrm{ap}\,(Y, X) := \varprojlim_{K_\alpha \hookrightarrow Y} \mathbb{M}\mathrm{ap}\,(K_\alpha, X)$$

is a well defined exponential object (with the correct universal property).

The story starts the same for topological stacks:

Let Y be as above and let $\mathscr{X}$ be a topological stack. Then:

- $\mathbb{M}\mathrm{ap}\,(K_\alpha, \mathscr{X})$ is a topological stack for each compact subset $K_\alpha \subset Y$.

Would <u>like</u> to say:

$$\mathbb{M}\mathrm{ap}\,(Y,\mathscr{X}) := \underset{K_\alpha \hookrightarrow Y}{\underleftarrow{\mathrm{holim}}}\, \mathbb{M}\mathrm{ap}\,(K_\alpha,\mathscr{X}) \ \text{ (weak 2 - limit)}$$

<u>But</u>

- This weak 2-limit may not exist
- $Y = \varinjlim\limits_{K_\alpha \hookrightarrow Y} K_\alpha$ in $\mathbb{CGH}$ does not imply $Y = \underset{K_\alpha \hookrightarrow Y}{\underrightarrow{\mathrm{holim}}}\, K_\alpha$ in $\mathfrak{TSt}$
  (The Yoneda embedding does not preserve arbitrary colimits)

VI. The Compactly Generated Grothendieck Topology

<u>Recall:</u>
Let $(\mathscr{C}, J)$ be a Grothendieck site.

What colimits <u>does</u> the Yoneda embedding $y : \mathscr{C} \hookrightarrow Sh_J(\mathscr{C})$ preserve?

- If $\left(C_\alpha \xrightarrow{f_\alpha} C\right)$ is a J-cover,

$$C = \varinjlim_{C_\alpha \to C} C_\alpha \ \text{ (in } Sh_J(\mathscr{C})\text{)}.$$

<u>Idea:</u> Want a Grothendieck topology $\mathscr{CG}$ on $\mathbb{CGH}$ such that for all Y, $(K_\alpha \hookrightarrow Y)$-the inclusion of all compact subsets- is a $\mathscr{CG}$-cover.

Theorem 1. *(Factorization Theorem of Topos Theory) Let $\mathscr{C}$ be a small category, then there is a bijection:*

$$\left\{ \begin{matrix} \textit{Grothendieck Topologies} \\ J \textit{ on } \mathscr{C} \end{matrix} \right\} \overset{1:1}{\leftrightarrow} \left\{ \begin{matrix} \mathscr{E} \rightarrowtail \mathrm{Set}^{\mathscr{C}^{op}} \\ \textit{geometric embedding of topois} \end{matrix} \right\}$$

given by assigning J to the left-exact adjoint pair

$$Sh_J(\mathscr{C}) \underset{i}{\overset{Sh_J}{\leftrightarrows}} \mathrm{Set}^{\mathscr{C}^{op}} ,$$

which constitutes a geometric embedding of the topos of sheaves into the topos of presheaves.

Let $\mathbb{CH}$ =compact Hausdorff spaces. Consider the full and faithful embedding

$$j : \mathbb{CH} \hookrightarrow \mathbb{CGH} \rightsquigarrow$$

$$\mathrm{Set}^{\mathbb{CH}^{op}} \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathrm{Set}^{\mathbb{CGH}^{op}} ,$$

which constitutes a geometric embedding $\mathrm{Set}^{\mathbb{CH}^{op}} \rightarrowtail \mathrm{Set}^{\mathbb{CGH}^{op}}$.

We define $Sh(\mathbb{CH})$ as the unique topos fitting in the following pullback diagram:

$$\begin{array}{ccc} Sh\,(\mathbb{CH}) & \overset{\alpha}{\rightarrowtail} & Sh\,(\mathbb{CGH}) \\ \downarrow & & \downarrow \\ \mathrm{Set}^{\mathbb{CH}^{op}} & \rightarrowtail & \mathrm{Set}^{\mathbb{CGH}^{op}} \end{array}$$

$\mathscr{CG}$ is the Grothendieck topology on $\mathbb{CGH}$ corresponding to the composite

$$Sh\,(\mathbb{CH}) \rightarrowtail \mathrm{Set}^{\mathbb{CH}^{op}} \rightarrowtail \mathrm{Set}^{\mathbb{CGH}^{op}}.$$

Properties of $\mathscr{CG}$:

i) α is an embedding $\Rightarrow$ every open cover is a $\mathscr{CG}$-cover

ii) $(K_\alpha \hookrightarrow Y)$ is a $\mathscr{CG}$-cover

iii) Every $\mathscr{CG}$-cover of a locally compact space X can be refined by an open one.

Definition 2. A **compactly generated stack** is an $\mathscr{X} \in \mathrm{St}_{\mathscr{CG}}\,(\mathbb{CGH})$ such that

$$\mathscr{X} \cong \mathrm{St}_{\mathscr{CG}}\,(\tilde{y}\,(\mathcal{G})) =: [\mathcal{G}]_{\mathscr{CG}}$$

for some topological groupoid $\mathcal{G}$.

Remark. $[\mathcal{G}]_{\mathscr{CG}} \cong \mathrm{St}_{\mathscr{CG}}\,([\mathcal{G}])$ and $[\mathcal{G}]_{\mathscr{CG}}\,(T) \cong \mathrm{Bun}_{\mathcal{G}}^{\mathscr{CG}}\,(T)$.

VII. Sketch of Main Results

i) $\Rightarrow \mathscr{CG}\mathfrak{TSt} \subset \mathrm{St}\,(\mathbb{CGH})$ (open cover topology)

ii) $\Rightarrow Y = \underset{K_\alpha \hookrightarrow Y}{\underrightarrow{\mathrm{holim}}}\, K_\alpha$ in $\mathscr{CG}\mathfrak{TSt}$.

iii) $\Rightarrow [\mathcal{G}]$ and $[\mathcal{G}]_{\mathscr{CG}}$ agree on locally compact spaces.

ii) + iii) $\Rightarrow \mathrm{St}_{\mathscr{CG}}$ restricts to an equivalence

$$\{\mathfrak{TSt} \text{ with a locally compact atlas}\} \xrightarrow{\sim} \mathscr{CG}\mathfrak{TSt}$$

iii) $\Rightarrow$ fully faithful (in a 2-categorical sense) and ii) $\Rightarrow$ essentially surjective:
If $Y \twoheadrightarrow \mathscr{Y}$ is any atlas for $\mathscr{Y}$ in $\mathscr{CG}\mathfrak{TSt}$, then $\coprod K_\alpha \twoheadrightarrow Y \twoheadrightarrow \mathscr{Y}$ is a locally compact atlas.

A. iii) + Noohi $\Rightarrow$ For every compact space K and $\mathscr{X} \in \mathscr{CG}\mathfrak{TSt}$, $\mathbb{M}\mathrm{ap}\,(K, \mathscr{X}) \in \mathscr{CG}\mathfrak{TSt}$.

B. iii) + Results of D. Gepner and A. Henriques $\Rightarrow \mathscr{CG}\mathfrak{TSt}$ has all weak limits.

C. A. + B. + ii) $\Rightarrow \mathbb{M}\mathrm{ap}\,(Y, \mathscr{X}) := \underset{K_\alpha \hookrightarrow Y}{\underleftarrow{\mathrm{holim}}}\, \mathbb{M}\mathrm{ap}\,(K_\alpha, \mathscr{X})$ (weak 2 - limit in $\mathscr{CG}\mathfrak{TSt}$) exists.

D. C. + formal nonsense $\Rightarrow \mathbb{M}\mathrm{ap}\,(\mathscr{Y}, \mathscr{X}) \in \mathscr{CG}\mathfrak{TSt}$ for all $\mathscr{X}$ and $\mathscr{Y}$ in $\mathscr{CG}\mathfrak{TSt}$.

Therefore compactly generated stacks are both Cartesian closed and complete.