Counting tropical plane curves

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Tropical curves
Counting curves
Lattice paths
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Evaluation maps

Talk 1

Tropical plane curves and lattice paths

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Idea of tropical geometry

- Replace algebraic varieties by degenerations, tropical varieties, which are piece-wise linear objects.
- The tropical varieties can be studied using combinatorics and linear algebra methods.
- Use the tropical varieties to prove theorems about algebraic varieties.

Why degenerations?

K the field of Puiseux-series.

$$K = \{\alpha(t) = a_1 t^{q_1} + a_2 t^{q_2} + \ldots\}$$

$a_i \in \mathbb{C}$, $q_1 < q_2 < \ldots \in \mathbb{Q}$ sharing a common denominator.

Valuation $\operatorname{val} : K^* \to \mathbb{R} : \alpha(t) \mapsto q_1$.

For a curve $V \subset (K^*)^2$ define

$$\operatorname{Val}(V) := \overline{\{(-\operatorname{val} x, -\operatorname{val} y), (x,y) \in V\}} \subset \mathbb{R}^2$$

the Tropicalization of V.

Why piece-wise linear?

Let $f = \sum a_{ij} x^i y^j \in K[x, y]$.

Definition

$\mathrm{trop}(f) := \max\{-\mathrm{val}(a_{ij}) + ix + jy\}$.

Definition

$\mathrm{Trop}(f) :=$

$\{(x, y) \in \mathbb{R}^2$, the maximum $\mathrm{trop}(f)$ is attained at least twice at $(x, y)\}$

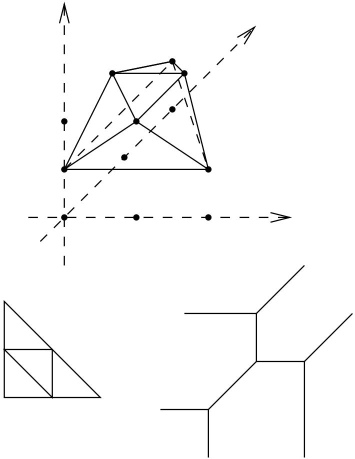
=Corner locus of the piece-wise linear map

$\mathrm{trop}(f) : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto \max\{-\mathrm{val}(a_{ij}) + ix + jy\}$.

Theorem (Kapranov)

$\mathrm{Val}(V(f)) = \mathrm{Trop}(f)$.

Dual Newton subdivisions



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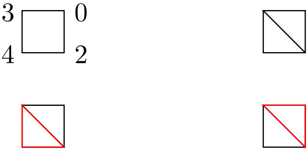
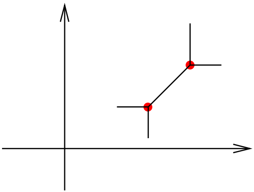
How to draw tropical curves

$f = 2t^{-4} + t^{-2}x + (t^{-3} + 3t)y + xy$

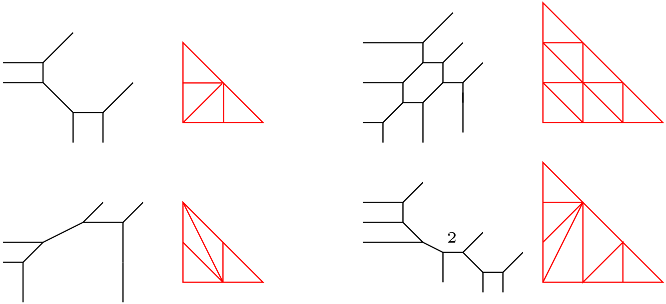
$\mathrm{trop}(f) = \max\{4, 2 + x, 3 + y, x + y\}$



$4 = 2 + x = 3 + y$ $\qquad$ $2 + x = 3 + y = x + y$



More dual Newton subdivisions



Combinatorial definition

Fact: Tropicalizations of Puiseux-series curves are piece-wise linear graphs in the plane satisfying the balancing condition. Now we can use this as a definition:

Definition (Roughly)

Tropical curves are weighted graphs satisfying the balancing condition.

Problem: are all those combinatorial objects tropicalizations of algebraic curves?

- True for plane curves of any genus (Speyer, Mikhalkin).
- True for rational curves in $\mathbb{R}^n$ for any n (Nishinou/Siebert).
- Not true in general.

Enumerative geometry

- Choose some *geometric objects* (e.g. curves),
- require them to satisfy certain *conditions* (e.g. pass through a given point, have degree d, have genus g...),
- and *count* how many objects satisfy the conditions.

The conditions have to be chosen in such a way that a finite number of the objects satisfy them.

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Counting plane curves

Definition

Let $N(d,g)$ denote the number of nodal plane curves of degree d and genus g (or equivalently, with $\binom{d-1}{2} - g$ nodes) passing through $3d+g-1$ points in general position.

$N(1,0) = 1$, $N(2,0) = 1$, $N(3,0) = 12$, ...

Tropical translations:

- nodal $\equiv$ only 3- and 4-valent vertices ("crossings").
- degree d $\equiv$ dual to a subdivision of the triangle $\Delta_d = \text{conv}((0,0),(0,d),(d,0))$.
- genus g $\equiv$ the graph has genus g (4-valent vertices are crossings).

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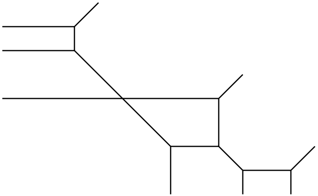
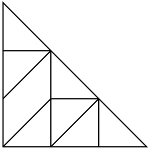
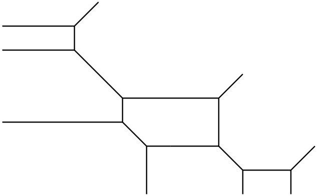
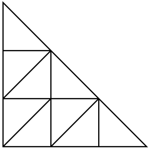
A nodal rational $(g = 0)$ tropical curve of degree 3:





A nodal elliptic $(g = 1)$ tropical curve of degree 3:





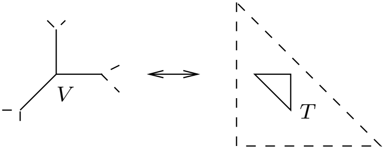
Counting tropical curves

Definition

Let $N^{\text{trop}}(d,g)$ denote the number of tropical nodal plane curves of degree d and genus g passing through $3d+g-1$ points in general position.

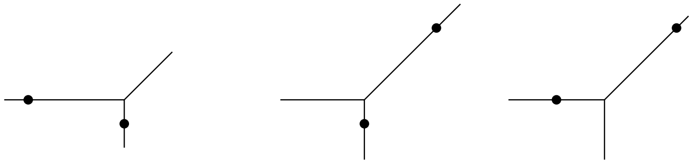
- tropical curves have to be counted with multiplicity
- mult $\equiv$ number of algebraic curves that degenerate to the tropical curve
- multiplicity can be defined combinatorially:

$\text{mult}(C) := \prod_V \text{mult}(V)$, $\text{mult}(V) := 2 \cdot \text{Area}(T)$.



Mikhalkin's Correspondence Theorem

$N^{\text{trop}}(1,0) = 1$:

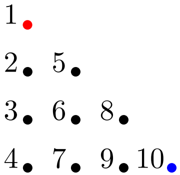


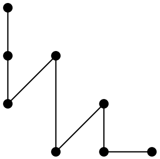
Theorem

$$N(d,g) = N^{\text{trop}}(d,g).$$

Lattice paths

Paths from top left (red) to bottom right (blue) respecting the order:





Positive multiplicity of a path γ

Defined recursively:

- Find the first left turn of a path.
- Define two new paths γ' and γ'' by:
  - γ': Cut the corner (triangle T).
  - γ'': Complete the corner to a parallelogram.
- $\text{mult}_+(\gamma) = 2\,\text{Area}(T) \cdot \text{mult}_+(\gamma') + \text{mult}_+(\gamma'')$.
- mult of any path which does not fit into Δ_d is 0.
- mult of the path on the top boundary with all the lattice points is 1.

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Multiplicity of a path γ

Negative multiplicity $\text{mult}_-(\gamma)$ defined analogously with the first right turn.

$\text{mult}(\gamma) = \text{mult}_+(\gamma) \cdot \text{mult}_-(\gamma)$.

Theorem (Mikhalkin)

$N^{\text{trop}}(d,g)$ equals the number of lattice paths in Δ_d with $3d+g-1$ steps counted with multiplicity.

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Moduli spaces of tropical curves

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Abstract tropical curves

Definition

An abstract tropical curve of genus g with n markings is a connected graph of genus g, such that

- each vertex is at least 3-valent,
- n of the leaves (ends, unbounded edges) are marked and
- the bounded edges are equipped with a (positive) length.

$M_{\mathrm{trop},g,N}$ = space of abstract tropical curves with N ends, all marked.

combinatorial type = graph with markings, without the lengths

For a type α, the subset $M^{\alpha}_{\mathrm{trop},g,N}$ of curves of this type is a positive orthant.

Parametrized tropical curves

Definition

(Γ, x_i, h) is an *n-marked parametrized tropical curve* of degree d and genus g to $\mathbb{R}^2$ if:
(Γ, x_i) abstract tropical curve of genus g with $N = n + 3d$ ends,
$h : \Gamma \to \mathbb{R}^2$ a continuous map satisfying:

- On each edge E, h is of the form

  $$h|_E : [0, l(E)] \to \mathbb{R}^2 : t \mapsto a + v(E) \cdot t$$

  ($a \in \mathbb{R}^2$, $v(E) \in \mathbb{Z}^2$ "direction" of E). ($v(E)$ = product of the weight $\omega(E)$ and the primitive integral vector.)
- For every vertex the balancing condition holds.
- Marked ends x_i contracted to a point by h (i.e. $v(x_i) = 0$).
- d of the other ends map to $(-1, 0)$, d to $(0, -1)$, d to $(1, 1)$.

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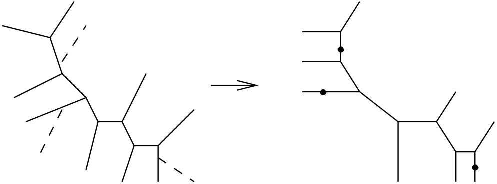

The length of an image edge $h(E)$ is determined by the length of E in the abstract tropical curve and the direction $v(E)$ (if $v(E) \neq 0$).

The space of all n-marked parametrized tropical curve of degree d and genus g to $\mathbb{R}^2$ is denoted by $M_{\text{trop},g,n}(\mathbb{R}^2, d)$.

combinatorial type: combinatorial type of (Γ, x_i) plus all direction vectors.

Evaluation maps

Definition

The map

$$\begin{aligned} \mathrm{ev}_i : M_{\mathrm{trop},g,n}(\mathbb{R}^2, d) &\to \mathbb{R}^2 \\ (\Gamma, x_1, \dots x_n, h) &\longmapsto h(x_i) \end{aligned}$$

is called the i-th evaluation map.

Lemma

The i-th evaluation map ev_i is linear on each cell M^α of $M_{\mathrm{trop},g,n}(\mathbb{R}^2, d)$.

ev-Multiplicity

Let $n = 3d + g - 1$. Then

$$\mathrm{ev} := \mathrm{ev}_1 \times \ldots \times \mathrm{ev}_n : M_{\mathrm{trop},g,n}(\mathbb{R}^2, d) \to \mathbb{R}^{2n}$$

is a map between polyhedral complexes of the same dimension.

Definition

For $C = (\Gamma, x_i, h)$, define $\mathrm{mult}_{\mathrm{ev}}(C)$ as the weight of the cell of C times the absolute value of the determinant of the square matrix ev.

Fact (lattice index computation): $\mathrm{mult}_{\mathrm{ev}}(C)$ equals the absolute value of the determinant of the map ev times the loop map, starting from the surrounding vector space $\mathbb{R}^{2+\#\text{ bounded edges}}$.

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The enumerative problem

For $P = (p_1, \dots, p_n) \in \mathbb{R}^{2n}$ define

Definition

$$\deg_{\mathrm{ev}}(P) := \sum_{C \in \mathrm{ev}^{-1}(P)} \mathrm{mult}_{\mathrm{ev}}(C).$$

Theorem (Gathmann/M)

- $\deg_{\mathrm{ev}}$ *is constant, i.e. does not depend on the choice of general* P.
- $\deg_{\mathrm{ev}} = N^{\mathrm{trop}}(d, g)$.

THANK YOU!